TOKAI UNIVERSITY

The cut locus of a certain class of cylinders

by

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Contents

Chapter 1

Introduction

Poincaré first introduced the notion of the cut locus in [Po]. He investigated the structure of the cut locus of a point on a complete, simply connected and real analytic 2-dimensional Riemannian manifold. Thirty years after his research, Myers and Whitehead investigated the detail structures of the cut locus for 2-dimensional Riemannian manifolds. Myers proved that the cut locus of a point on an analytic 2-dimensional compact Riemannian manifold is a finite graph. Whitehead proved that the distance function to the cut locus of a point is continuous for any dimensional complete Riemannian manifolds. In 1994, Hebda [He] proved that the cut locus of a point in a complete 2-dimensional Riemannian manifold admits a local tree structure. His result was generalized by Shiohama and Tanaka [ShT] to the cut locus of a compact subset in an Alexandrov surface.

It is very difficult to determine the structure of the cut locus of a Riemannian manifold. Since Elerath succeeded in revealing the structure of the cut locus for paraboloids of revolution and 2-sheeted hyperboloids of revolution in 1980, the structure for quadric surfaces of revolution have been studied. After his work, the structure of the cut locus has been determined for quadric surfaces and the standard tori in Euclidean space by Itoh, Kiyohara, Sinclair and Tanaka.

In the present thesis, the structure of the cut locus is determined for a class of cylinders of revolution.

Chapter 2

Riemannian geometry

The curvature is one of concepts in Differential geometry. There are so many kinds of curvatures such as principal curvatures, mean curvatures, sectional curvatures, and Gaussian curvatures, etc. In this work, we are interesting in the Gaussian curvature. In this chapter we reach the definition of a Riemannian manifold by introducing classical surfaces of revolution first.

2.1 Surfaces of revolution

A surface of revolution S is a surface in Euclidean space obtained by rotating a plane curve in E^3 where the rotation is about a line that does not intersect the curve and is contained in the plane containing the curve. The line is called the *axis of rotation* and the curve is called the *generating curve* or the 0-*meridian*. Without loss of generality, we may assume that the curve is a unit speed xz-plane curve and the axis of revolution is the z-axis.

Let $c(t) = (m(t), 0, z(t))$, where $m(t) > 0$, be a unit speed xz-plane curve without self intersection. Thus $\dot{m}(t) + \dot{z}(t) = 1$. Then the surface of revolution S which is generated by this plane curve in parametric form is

$$\mathbf{x}(t, \theta) = (m(t)\cos\theta, m(t)\sin\theta, z(t)), \quad t \in R^2, 0 \leqslant \theta < 2\pi.$$

The curves on the surface of revolution obtained by holding θ constant and varying t are called *meridians* or *longitudes*, and the curves on the surface obtained by holding t constant and varying θ are called *circle of latitudes* or *parallel.* The meridian opposite 0-meridian is called π-*meridian* (Figure 2.1).

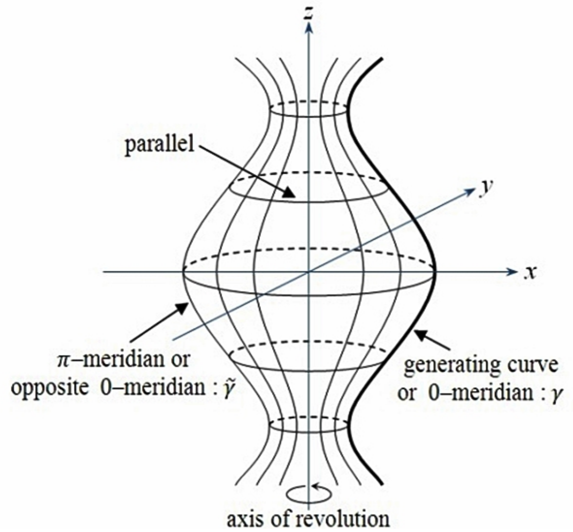

FIGURE 2.1: Surface of revolution

The meridian in parametric form is $\mathbf{x}(t, \theta_0) = (m(t)\cos\theta_0, m(t)\sin\theta_0, z(t))$, where θ_0 is a constant. The parallel in parametric form is $\mathbf{x}(t_0, \theta) = (m(t_0)\cos\theta, m(t_0)\sin\theta, z(t_0))$, where t_0 is a constant.

Examples of a surface of revolution

1. A sphere of radius R is obtained by rotating a semicircle of radius R centered at the origin. A typical parametric form is given by

$$\mathbf{x}(t, \theta) = (R\cos t\cos\theta, R\cos t\sin\theta, R\sin t).$$

2. A right circular cylinder is obtained by rotating a straight line parallel to the z-axis with constant distance a from the z-axis has a parametric in the form

$$\mathbf{x}(t, \theta) = (a\cos\theta, a\sin\theta, t).$$

A surface S is called a *complete* if every Cauchy sequence of points of S converges on S. It is well known that Euclidean space E^3 is complete, that is, every Cauchy sequence of points of E^3 converges to a point. Moreover, any closed subset S of E^3 is complete, that is, any Cauchy sequence of points in S is also a Cauchy sequence of any point in E^3, which has a limit point p. Since S is closed, $p \in S$ and hence S is complete.

One of the most widely studied geometrical objects is a Riemannian manifold. A surface in E^3 is a 2-dimensional Riemannian manifold. The plane is represented by R^2, while R^1 denotes a real line.

The unit circle in R^2 is denoted by S^1. We denote a unit circle $\{x \in R^2 | ||x|| = 1\}$ by S^1, and thus $S^1 \times R^1 \subset R^2 \times R^1 = R^3$ is a right circular cylinder.

A subset $S \subset R^3$ is said to be *connected* if it cannot be expressed as a union of two non-empty disjoint subsets each of which is open in S.

2.2 Tangent planes

A subset $S \subset R^3$ is called a *regular surface* if, for each $p \in S$, there exists a neighborhood V in R^3 and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset R^2$ onto $V \cap S \subset R^3$ such that

1. $\mathbf{x}$ is differentiable. This means that if we write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U$$

   the functions $x(u, v), y(u, v), z(u, v)$ have continuous partial derivatives of all order in U.

2. $\mathbf{x}$ is a homeomorphism. Since $\mathbf{x}$ is continuous by condition 1, $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous; that is $\mathbf{x}^{-1}$ is the restriction of a continuous map $f : W \subset R^3 \to R^2$ defined on an open set W containing $V \cap S$.

3. For each $q \in U$ the differential $d\mathbf{x}_q : R^2 \to R^3$ is injective.

The mapping $\mathbf{x}$ is called a *parametrization* or a *system of local coordinates* in a neighborhood of p. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood*.

Let $M \subset R^3$ be a regular surface and let $p \in M$ be a point. A vector v in R^3 is a *tangent vector* to M at p if there exists a curve $c : (-\epsilon, \epsilon) \to M$ for some number $\epsilon > 0$ such that $c(0) = p$ and $c'(0) = v$. The collection of all tangent vectors to M at p is denoted by T_pM, and is called the *tangent plane* to M at p (Figure 2.2). Therefore, a regular surface has the inner product on each tangent plane induced from Euclidean inner product. Higher dimensional regular surfaces are defined similarly. Such higher dimensional regular surfaces are called *differentiable manifolds*.

A *Riemannian manifold* is a differentiable manifold with a given Riemannian metric, a correspondence which associates to each point p of M an inner product $\langle \, , \, \rangle_p$ on the tangent space T_pM. The regular surfaces are typical examples of a 2-dimensional Riemannian manifold.

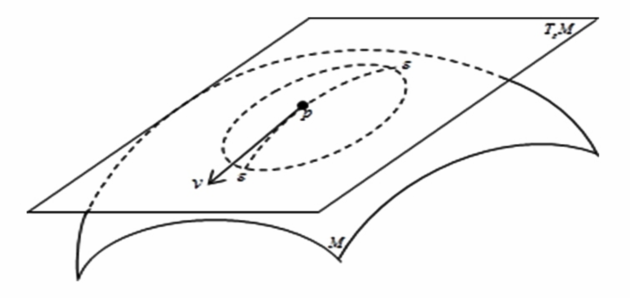

FIGURE 2.2: Tangent plane T_pM

2.3 Geodesics

A curve γ on a Riemannian manifold is called a *geodesic* if γ is locally minimizing. A geodesic is characterized by a system of ordinary differential equations. If M is a regular surface in E^3, a curve γ on M is a geodesic if and only if $\gamma''(s)$ is orthogonal to the tangent space $T_{\gamma(s)}M$ for all s.

Examples for geodesics of some regular surfaces in E^3

1. Geodesics of a plane

   Let $P = \{x \in R^3 | \langle x, a\rangle = b\}$ be a plane orthogonal to the unit vector $a \in E^3$. If $\gamma : [a, b] \to P$ is an arbitrary differentiable curve on P, we have $\langle \gamma(t), a\rangle = b$ for each $t \in [a, b]$. Thus $\langle \gamma''(t), a\rangle = 0$, that is,

   $$\gamma''(t) \in T_{\gamma(t)}P$$

   holds for any $t \in [a, b]$, where $T_{\gamma(t)}P$ denotes the tangent plane at $\gamma(t)$. Hence γ is a geodesic if and only if $\gamma'' = 0$, that is, $\gamma(t) = ct + d$ where $c, d \in E^3$. Therefore, we conclude that the geodesics of a plane are the straight lines parameterized proportionally to the arclength in the plane.

2. Geodesics of a sphere

   Let γ be a differentiable curve parameterized by arclength on the sphere $S^2(r)$ centered at a point $a \in R^3$ with radius $r > 0$. We have $|\gamma(t) - a|^2 = r^2$ for all t. By differentiating this expression two times, we obtain $\langle \gamma(t) - a, \gamma''(t)\rangle = -1$. Since the tangent plane $T_{\gamma(t)}S^2(r)$ is the orthogonal complement of the radius vector $\gamma(t) - a$, we may have

   $$[\gamma''(t)]^T = \gamma''(t) + \frac{1}{r^2}(\gamma(t) - a),$$

where $[\gamma''(t)]^T$ denotes the orthogonal projection of $\gamma''(t)$ to the tangent plane $T_{\gamma(t)}M$. Thus, γ is a geodesic if and only if γ satisfies the differential equation

$$r^2\gamma''(t) + \gamma(t) - a = 0$$

with the conditions $|\gamma(t) - a|^2 = r^2$ and $|\gamma'(t)|^2 = 1$. Here we have

$$\gamma(t) = a + p\cos\frac{t}{r} + rv\sin\frac{t}{r},$$

where $p = \gamma(0)$ and $v = \gamma'(0)$. Therefore, we conclude that the geodesics of a sphere are the great circles.

3. Geodesics of a flat cylinder

Suppose that $\gamma : R \to C$ be a unit speed geodesic on C defined by $\gamma(t) := (x(t), y(t), z(t))$ with $\gamma(0) = (1, 0, 0)$. Since $\gamma'(t)$ is a unit tangent vector, we obtain

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = 1. \tag{2.1}$$

Since the tangent plane of C at $\gamma(0)$ is parallel to the yz-plane, we may assume that

$$\gamma'(0) = (0, a, b) \tag{2.2}$$

for some $a, b \in R$. In particular, we have

$$y'(0) = a \quad \text{and} \quad z'(0) = b. \tag{2.3}$$

Since $\gamma''(t)$ is orthogonal to the tangent plane to C, $\gamma''(t)$ is parallel to $(x(t), y(t), 0)$ for each t. Therefore there exists a function $k(t)$ such that

$$x''(t) = k(t)x(t) \quad \text{and} \quad y''(t) = k(t)y(t). \tag{2.4}$$

Since $\gamma(t)$ is a curve on C, for each t

$$x(t)^2 + y(t)^2 = 1. \tag{2.5}$$

Since $z(0) = 0$ and $z'(0) = b$, we get

$$z(t) = bt. \tag{2.6}$$

By differentiating the equation (2.5) twice, we obtain

$$xx'' + yy'' + (x')^2 + (y')^2 = 0. \tag{2.7}$$

By combining (2.1), (2.3), (2.4), and (2.5), we have

$$k(t) = b^2 - 1 = -a^2.$$

From (2.4), we obtain differential equations

$$x''(t) = -a^2 x(t) \quad \text{and} \quad y''(t) = -a^2 y(t)$$

with initial conditions $x(0) = 1, x'(0) = 0$ and $y(0) = 0, y'(0) = a$, respectively. The solutions are well known, i.e., $x(t) = \cos at$ and $y(t) = \sin at$ and hence

$$\gamma(t) = (\cos at, \sin at, bt)$$

with $a^2 + b^2 = 1$. Therefore, we conclude that the geodesics of a right circular cylinder are circular helices.

2.4 Hopf-Rinow theorem

The following theorem, which is called the Hopf-Rinow theorem, is fundamental in Riemannian geometry.

Theorem 2.1. *Let M be a Riemannian manifold and let $p \in M$. The following assertions are equivalent:*

(a) $\exp_p$ *is defined on all of T_pM.*

(b) The closed and bounded sets of M are compact.

(c) M is complete as a metric space.

(d) M is geodesically complete.

(e) There exists a sequence of compact subsets $K_n \subset M, K_n \subset K_{n+1}$ and $\bigcup_n K_n = M$, such that if $q_n \notin K_n$ then $d(p, q_n) \to \infty$.

In addition, any of statements above implies that

(f) For any $q \in M$ there exists a geodesic γ joining p to q with $l(\gamma) = d(p, q)$.

The Hopf-Rinow theorem says that every geodesic on a complete connected manifold M can be extended indefinitely to both directions. For each pair $p, q \in M$, p and q can be joined by a minimal geodesic segment.

The geodesics on Euclidean plane are straight lines, and any two points p and q can be joined by a unique line segment.

2.5 Exponential map

Let p be a point on a complete connected Riemannian manifold M. The exponential map $\exp_p : T_pM \to M$, where T_pM is tangent plane of M at p, is defined by

$$\exp_p(v) = \gamma_v(1)$$

for all v in T_pM, where $\gamma_v : [0, \infty] \to M$ denotes the unit speed geodesic on M satisfying

$$\gamma_v(0) = p, \gamma_v'(0) = v.$$

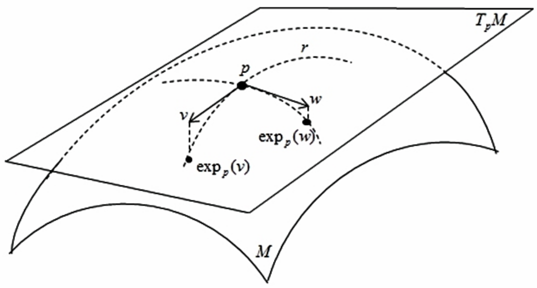

FIGURE 2.3: Exponential map

2.6 Curvature tensors

Let M denote a Riemannian manifold, and ∇ the Riemannian connection induced from its Riemannian metric. Then the curvature tensor R of M is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for vector fields X, Y, and Z on M.

Let σ denote a plane of a tangent space T_pM of p spanned by X and Y. Then the sectional curvature $K(\sigma)$ of σ is defined by

$$K(\sigma) := \frac{\langle R(X, Y)Y, X \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2}.$$

For a 2-dimensional Riemannian manifold, the sectional curvature is called the *Gaussian curvature.*

2.7 Jacobi fields

We consider a family of geodesics $\gamma_\tau : [0,1] \to M, \tau \in (-\epsilon, \epsilon)$. Put $f(t,\tau) := \exp_p(tv(\tau))$, where $v(\tau) = \gamma'_\tau(0)$. We get a vector field $J(t) := \frac{\partial f}{\partial \tau}(t,0)$ along the geodesic γ_0. Since γ_τ is a geodesic for each $\tau \in (-\epsilon, \epsilon)$, we get $\frac{D}{dt}\frac{\partial f}{\partial t} = 0$, where $\frac{D}{dt}$ denotes the covariant derivative along γ_τ for each fixed τ. By $\frac{D}{d\tau}$, we denote the covariant derivative along the τ-curve of f. Thus,

$$0 = \frac{D}{\partial \tau}\left(\frac{D}{\partial t}\frac{\partial f}{\partial t}\right) = \frac{D}{\partial t}\frac{D}{\partial \tau}\frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial \tau}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} = \frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial f}{\partial \tau} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial \tau}\right)\frac{\partial f}{\partial t}.$$

Since we put $\frac{\partial f}{\partial \tau}(t,0) = J(t)$,

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0,$$

for all $t \in [0,1]$, where R denotes the curvature tensor of M. Suppose that M is a 2-dimensional Riemannian manifold. Let w_0 denote a unit tangent vector at $\gamma(0)$ orthogonal to $\gamma'(0)$, where $\gamma := \gamma_0$. Let $w(t)$ denote the parallel vector filed along γ with $w(0) = w_0$. Then we obtain an orthogonal basis $w(t), \gamma'(t)$ for $T_{\gamma(t)}M$ for each $t \in [0,1]$.

Suppose that the Jacobi field $J(t)$ is orthogonal to $\gamma'(t)$ for each $t \in [0,1]$. Then we have

$$J(t) = \langle J(t), w(t)\rangle\, w(t).$$

If we put $\langle J(t), w(t)\rangle =: f(t)$, we get a differential equation

$$f''(t) + G(\gamma(t))f(t) = 0,$$

where $G(\gamma(t))$ denotes the Gaussian curvature at $\gamma(t)$.

2.8 Conjugate points and conjugate loci

Let $\gamma|_{[0,t_1]}$ be a geodesic on a complete connected Riemannian manifold M. The point $\gamma(t_0), t_0 \in (0, t_1]$ is said to be *conjugate* to $\gamma(0)$ along γ, if there exists a Jacobi field J along γ, not identically zero, with $J(0) = 0 = J(t_0)$. The *conjugate locus* of p is defined as the set of the conjugate points along all geodesics emanating from p. If $\gamma(t_0)$ is conjugate to $\gamma(0)$, then $\gamma(0)$ is conjugate to $\gamma(t_0)$.

For example, let γ be a geodesic starting at any point p of a 2-sphere S with radius r. Then the Jacobi equation is

$$\frac{d^2 f}{ds^2} + \frac{1}{r^2} f = 0,$$

where s is the arc length parameter of γ. We know that

$$f(s) = a \sin \frac{s}{r} + b \cos \frac{s}{r},$$

where a, b are arbitrary constants. Suppose the initial conditions are $f(0) = 0, f'(0) = 1$. Thus we obtain

$$f(s) = \frac{1}{r} \sin \frac{s}{r}.$$

Since $f(s) = 0$ if $\frac{s}{r} = n\pi$ or $s = n\pi r$, where $n = 1, 2, 3, \ldots$. The first conjugate point of $\gamma(0) = p$ on γ is the point $f(\pi r) = 0$, i.e., the antipodal point $-p$ of p.

2.9 Cut points and cut loci

Let $\gamma|_{[0,t_1]}$ be a unit speed minimal geodesic emanating from a point $p = \gamma(0)$ of a complete connected manifold M. If all geodesic extensions $\gamma|_{[0,t_2]}$ of γ are not minimal anymore, the endpoint $\gamma(t_1)$ is called a *cut point* of p along γ.

The *cut locus* of p is the set of all cut points along all minimal geodesics emanating from p and we denote the set by C_p.

2.10 The Sturm and Rauch comparison theorem

The following theorem is very useful to determine the structure of the cut locus.

Theorem 2.2 (Sturm comparison theorem)**.** *Let $x_1(t)$ and $x_2(t)$ be solutions to equations*

$$x_1''(t) + p_1(t)x_1(t) = 0 \tag{2.8}$$

and

$$x_2''(t) + p_2(t)x_2(t) = 0 \tag{2.9}$$

with the initial conditions $x_1(0) = x_2(0) = 0$ and $x_1'(0) = x_2'(0) = 1$, where $p_1(t)$ and $p_2(t)$ are continuous on $[0, T]$. Suppose $p_1(t) \leqslant p_2(t)$ on $[0, T]$ and $x_2(t) > 0$ on $(0, T]$. Then $x_1(t) \geqslant x_2(t)$ on $[0, T]$.

Proof. Mutiplying (2.8) by $x_2(t)$ and (2.9) by $x_1(t)$, and subtracting one from the other, we get

$$x_1(t)x_2''(t) - x_2(t)x_1''(t) + (p_2(t) - p_1(t))x_1(t)x_2(t) = 0. \tag{2.10}$$

Assume that t_1 the first zeros of $x_1(t)$ with $t_1 < T$. Then $x_1'(t_1) < 0$ and $x_2(t_1) > 0$.

By using the technique of integrate by parts on (2.10) over $[0, t_1]$, thus

$$\begin{aligned} 0 &= \int_0^t \big(x_1(t)x_2''(t) - x_2(t)x_1''(t) + (p_2(t) - p_1(t))x_1(t)x_2(t)\big)\, dt \\ &= (x_1(t)x_2'(t) - x_2(t)x_1'(t))\Big|_0^{t_1} + \int_0^{t_1} ((p_2(t) - p_1(t))x_1(t)x_2(t))\, dt \\ &= -x_2(t_1)x_1'(t_1) + \int_0^{t_1} ((p_2(t) - p_1(t))x_1(t)x_2(t))\, dt \end{aligned}$$

But $x_1'(t_1) < 0, x_2(t_1) > 0, p_1(t) \leqslant p_2(t)$ and $x_1(t), x_2(t) > 0$ on $(0, t_1)$ imply that

$$-x_2(t_1)x_1'(t_1) + \int_0^{t_1} ((p_2(t) - p_1(t))x_1(t)x_2(t))\, dt > 0 \tag{2.11}$$

which is a contradiction. Thus $x_1(t) > 0$ on $[0, T]$. Here we get from above that

$$x_1(t)x_2'(t) - x_2(t)x_1'(t) \leqslant 0,$$

that is,

$$\frac{x_1'(t)}{x_1(t)} \geqslant \frac{x_2'(t)}{x_2(t)}.$$

Therefore, for any small $\epsilon > 0$ and any $t \in [\epsilon, T]$, which implies

$$\frac{x_1(t)}{x_2(t)} \geqslant \frac{x_1(\epsilon)}{x_2(\epsilon)}.$$

Then, by applying the L'Hôpital rule, we get

$$1 = \lim_{\epsilon \to 0} \frac{x_1'(\epsilon)}{x_2'(\epsilon)} = \lim_{\epsilon \to 0} \frac{x_1(\epsilon)}{x_2(\epsilon)} \leqslant \lim_{\epsilon \to 0} \frac{x_1(t)}{x_2(t)} = \frac{x_1(t)}{x_2(t)}.$$

Thus $x_1(t) \geqslant x_2(t)$ on $[0, T]$. □

The following is a generalization of the Sturm comparison theorem.

Theorem 2.3 (Rauch comparison theorem). *Let $\gamma : [0, a] \to M^n$ and $\tilde{\gamma} : [0, a] \to \widetilde{M}^{n+k}, k \geqslant 0$, be geodesics with the same velocity (i.e., $|\gamma'(t)| = |\tilde{\gamma}'(t)|$), and let J and $\widetilde{J}$ be Jacobi fields along γ and $\tilde{\gamma}$, respectively, such that*

$$J(0) = \widetilde{J}(0) = 0, \quad \langle J'(0), \gamma'(0) \rangle = \left\langle \widetilde{J}'(0), \tilde{\gamma}'(0) \right\rangle, \quad |J'(0)| = |\widetilde{J}'(0)|.$$

Assume that $\tilde{\gamma}$ *does not have conjugate points on* $(0,a]$ *and that, for all* t *and all* $x \in T_{\gamma(t)}M, \tilde{x} \in T_{\tilde{\gamma}(t)}\widetilde{M}$*, we have*

$$\widetilde{K}(\tilde{x}, \tilde{\gamma}'(t)) \geqslant K(x, \gamma'(t)),$$

where $K(x,y)$ *denotes the sectional curvature with respect to the plane generated by* x *and* y*. Then*

$$|\widetilde{J}| \leqslant |J|.$$

In addition, if for some $t_0 \in (0,a]$*, we have* $|\widetilde{J}(t_0)| = |J(t_0)|$*, then* $\widetilde{K}(\widetilde{J}(t), \tilde{\gamma}'(t)) = K(J(t), \gamma'(t))$*, for all* $t \in [0, t_0]$*.*

Chapter 3

The case where the Gaussian curvature is decreasing on the upper half meridian

It is a very difficult problem to determine the structure of the cut locus of a Riemannian manifold and it was difficult even for a quadric surface.

Since Elerath ([E]) succeeded in specifying the structure of the cut locus for paraboloids of revolution and (2-sheeted) hyperboloids of revolution, the structures of the cut locus for quadric surfaces of revolution have been studied. After his work, Sinclair and Tanaka ([ST]) determined the structure of the cut locus for a class of surfaces of revolution containing the ellipsoids. Notice that the structures of the cut locus for triaxial ellipsoids with unequal axes were also determined by Itoh and Kiyohara ([IK]).

On the structure of the cut locus for a cylinder of revolution $(R^1 \times S^1, dt^2 + m(t)^2 d\theta^2)$, Tsuji ([Ts]) first determined the cut locus of a point on the equator $t = 0$ if the cylinder is symmetric with respect to the equator and the Gaussian curvature is decreasing on the upper half meridian $t > 0, \theta = 0$. In 2003, Tamura ([Ta]) determined the structure of the cut locus by adding an assumption $m' \neq 0$ except $t = 0$. In this chapter, we determine the structure of the cut locus without this assumption.

Here, let us review the notion of a cut point and the cut locus of a point. Let $\gamma : [0, a] \to M$ be a minimal geodesic segment in a complete Riemannian manifold M. The end point of $\gamma(a)$ is called a *cut point* of $\gamma(0)$ along γ, if any geodesic extension of γ is not minimal anymore. The *cut locus* C_p of a point p of M is by definition the set of the cut points along all minimal geodesic segments emanating from p.

In this chapter we will prove the following theorem.

Main Theorem. *Let (M, ds^2) be a complete Riemannian manifold $R^1 \times S^1$ with a warped product metric $ds^2 = dt^2 + m(t)^2 d\theta^2$ of the real line (R^1, dt^2) and the unit circle $(S^1, d\theta^2)$. Suppose that the warping function m is a positive-valued even function and the Gaussian curvature of M is decreasing along the half meridian $t^{-1}[0,\infty) \cap \theta^{-1}(0)$. If the Gaussian curvature of M is positive on $t = 0$, then the structure of the cut locus C_q of a point $q \in \theta^{-1}(0)$ in M is given as follows:*

1. *The cut locus C_q is the union of a subarc of the parallel $t = -t(q)$ opposite to q and the meridian opposite to q if $|t(q)| < t_0 := \sup\{t > 0 \mid m'(t) < 0\}$ and $\varphi(m(t(q))) < \pi$. More precisely,*
$$C_q = \theta^{-1}(\pi) \cup \left(t^{-1}(-t(q)) \cap \theta^{-1}[\varphi(m(t(q))), 2\pi - \varphi(m(t(q)))]\right).$$
2. *The cut locus C_q is the meridian $\theta^{-1}(\pi)$ opposite to q if $\varphi(m(t(q))) \geq \pi$ or if $|t(q)| \geq t_0$.*

Here, the function $\varphi(\nu)$ on $(\inf m, m(0))$ is defined as
$$\varphi(\nu) := 2\int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2-\nu^2}}dt = 2\int_{0}^{\xi(\nu)} \frac{\nu}{m\sqrt{m^2-\nu^2}}dt,$$
where $\xi(\nu) := \min\{t > 0 \,|\, m(t) = \nu\}$. Notice that the point q is an arbitrarily given point if the coordinates (t, θ) are chosen so as to satisfy $\theta(q) = 0$.

Remark 3.1. If the Gaussian curvature of a cylinder of revolution is nonpositive everywhere, then any geodesic has no conjugate point. Therefore, it is clear to see that the cut locus of a point on the manifold is the meridian opposite to the point.

3.1 Preliminaries

Let f be the solution of the differential equation
$$f'' + Kf = 0 \tag{3.1}$$
with initial conditions $f(0) = c$ and $f'(0) = 0$. Here c denotes a fixed positive number and $K : [0,\infty) \to R$ denotes a continuous function.

Lemma 3.2. *If $K(0) > 0$ and $f'(t) \neq 0$ for any $t > 0$, then $f'(t) < 0$ on $(0,\infty)$. Furthermore, if $f > 0$ on $[0,\infty)$, then $K(t) < 0$ for some $t > 0$.*

Proof. Since $f''(0) = -K(0)f(0) < 0$ by (3.1), $f'(t)$ is strictly decreasing on $(0,\delta)$ for some $\delta > 0$. This implies that $0 = f'(0) > f'(t)$ for any $t \in (0,\delta)$. Since $f' \neq 0$ on $[0,\infty)$, $f'(t) < 0$ on $(0,\infty)$. Furthermore, we assume that $f > 0$ on $[0,\infty)$. Supposing that $K \geq 0$ on $[0,\infty)$, we will get a contradiction. By (3.1),

$$f''(t) = -K(t)f(t) \leq 0$$

on $[0,\infty)$. Hence $f'(t)$ is decreasing on $[0,\infty)$. In particular, $0 = f'(0) > f'(\delta) \geq f'(t)$ for any $t \geq \delta$. This contradicts the assumption $f > 0$. □

Lemma 3.3. *Suppose that $K(0) > 0$ and $f > 0$ on $[0,\infty)$. If $f'(t) = 0$ for some $t > 0$ and K is decreasing, then there exists a unique solution $t = t_0 \in (0,\infty)$ of $f'(t) = 0$ such that $f'(t) < 0$ on $(0,t_0)$ and $f'(t) > 0$ on (t_0,∞) and there exists $t_1 \in (0,t_0)$ satisfying $K(t_1) = 0$. Hence $K \geq 0$ on $[0,t_1]$ and $K \leq 0$ on $[t_1,\infty)$.*

Proof. Let $a > 0$ denote the minimum positive solution $t = a$ of $f'(t) = 0$. Suppose that there exist another solution $b(> a)$ satisfying $f'(b) = 0$. By the mean value theorem, there exist $t_1 \in (0,a)$ and $s_1 \in (a,b)$ satisfying $f''(t_1) = f''(s_1) = 0$. Hence $K(t_1) = K(s_1) = 0$ by (3.1). Since K is decreasing, $K = 0$ on $[t_1,s_1]$. Therefore, by (3.1), $f''(t) = 0$ on $[t_1,s_1]$. In particular, $f'(a) = f'(t_1) = 0$. Since $0 < t_1 < a$, t_1 is a positive solution t of $f'(t) = 0$, which is less than a. This is a contradiction. Therefore, there exists a unique positive solution $t = t_0$ of $f'(t) = 0$. From the mean value theorem and (3.1), there exists $t_1 \in (0,t_0)$ satisfying $K(t_1) = 0$. Since $K(t)$ is decreasing, $K \geq 0$ on $[0,t_1]$ and $K \leq 0$ on $[t_1,\infty)$. Hence by (3.1), $f''(t) = -K(t)f(t) \geq 0$ on $[t_1,\infty)$ and $f'(t) \geq f'(t_0) = 0$ for any $t > t_0$. Since f' has a unique positive zero, $f' > 0$ on (t_0,∞). It is clear from the proof of Lemma 3.2 that $f' < 0$ on $(0,t_0)$. □

3.2 Review of the behavior of geodesics

From now on, M denotes a complete Riemannian manifold $R^1 \times S^1$ with a warped product Riemannian metric $ds^2 = dt^2 + m(t)^2 d\theta^2$ of the real line (R^1, dt^2) and the unit circle $(S^1, d\theta^2)$. Let us review the behavior of a geodesic $\gamma(s) = (t(s),\theta(s))$ on the manifold M. For each unit speed geodesic $\gamma(s) = (t(s),\theta(s))$, there exists a constant ν satisfying

$$m(t(s))^2\theta'(s) = \nu. \tag{3.2}$$

Hence, if $\eta(s)$ denotes the angle made by the velocity vector $\gamma'(s)$ of the geodesic $\gamma(s)$ and the tangent vector $(\partial/\partial\theta)_{\gamma(s)}$, then

$$m(t(s))\cos\eta(s) = \nu \tag{3.3}$$

for any s. The constant ν is called the *Clairaut constant* of γ. The reader should refer to Chapter 7 in [SST] for the Clairaut relation. Since $\gamma(s)$ is unit speed,

$$t'(s)^2 + m(t(s))^2\theta'(s)^2 = 1 \tag{3.4}$$

holds. By (3.2) and (3.4), it follows that

$$t'(s) = \pm\frac{\sqrt{m(t(s))^2 - \nu^2}}{m(t(s))} \tag{3.5}$$

$$\theta(s_2) - \theta(s_1) = \varepsilon(t'(s))\int_{t(s_1)}^{t(s_2)} \frac{\nu}{m\sqrt{m^2 - \nu^2}}dt \tag{3.6}$$

holds, if $t'(s) \neq 0$ on (s_1, s_2) and $\varepsilon(t'(s))$ denotes the sign of $t'(s)$.

The length $L(\gamma)$ of a geodesic segment $\gamma(s) = (t(s), \theta(s))$, $s_1 \leq s \leq s_2$ is

$$L(\gamma) = \varepsilon(t'(s))\int_{t(s_1)}^{t(s_2)} \frac{m(t)}{\sqrt{m(t)^2 - \nu^2}}dt \tag{3.7}$$

if $t'(s) \neq 0$ on (s_1, s_2).

From a direct computation, the Gaussian curvature G of M is given by

$$G(q) = -\frac{m''}{m}(t(q))$$

at each point $q \in M$. Since G is constant on $t^{-1}(a)$ for each $a \in R$, a smooth function K on R is defined by

$$K(u) := G(q)$$

for $q \in t^{-1}(u)$. Therefore m satisfies the following differential equation

$$m'' + Km = 0$$

with $m'(0) = 0$.

From now on, we assume that the Gaussian curvature G of M is positive on $t^{-1}(0)$, and $m(t) = m(-t)$ holds for any $t \in R$. Hence, M is symmetric with respect to the equator $t = 0$ and if K is decreasing on $[0, \infty)$, then by Lemma 3.3, $m'(t) < 0$ for all $t > 0$ or there exists a unique positive solution $t = t_0$ of $m'(t) = 0$ such that $m' < 0$ on $(0, t_0)$ and $m' > 0$ on (t_0, ∞). Furthermore, if the latter case happens, there exists $t_1 \in (0, t_0)$ such that $K \geq 0$ on $[0, t_1]$ and $K \leq 0$ on $[t_1, \infty)$.

For technical reasons, we treat both geodesics on M and its universal covering space $\pi : \widetilde{M} \to M$, where $\widetilde{M} := (R^1 \times R^1, d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2)$.

Choose any point p on the equator $t = 0$. We may assume that $\theta(p) = 0$ without loss of generality. Let $\gamma : [0, \infty) \to M$ denote a geodesic emanating from $p = \gamma(0)$ with Clairaut constant $\nu \in (\inf m, m(0))$. Notice that γ is uniquely determined up to the reflection with respect to $t = 0$. The geodesic $\gamma(s) = (t(s), \theta(s))$ is tangent to the parallel $t = \xi(\nu)$ (if $(t \circ \gamma)'(0) > 0$) or $t = -\xi(\nu)$ (if $(t \circ \gamma)'(0) < 0$), where $\xi(\nu) > 0$ denotes the least positive solution of $m(\xi(\nu)) = \nu$, that is,

$$\xi(\nu) := \min\{u > 0 \mid m(u) = \nu\}.$$

After γ is tangent to the parallel $t = \xi(\nu)$ or $-\xi(\nu)$, γ intersects the equator $t = 0$ again. Thus, after $\tilde{\gamma}$ is tangent to the *parallel arc* $\tilde{t} = \xi(\nu)$ or $-\xi(\nu)$, $\tilde{\gamma}$ intersect $\tilde{t} = 0$ again. Here $\tilde{\gamma}$ denotes a geodesic on $\widetilde{M}$ satisfying $\gamma = \pi \circ \tilde{\gamma}$.

From (3.6), we obtain,

$$\tilde{\theta}(s_0) - \tilde{\theta}(0) = \int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = \int_{0}^{\xi(\nu)} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt,$$

and

$$\tilde{\theta}(s_1) - \tilde{\theta}(s_0) = \int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = \int_{0}^{\xi(\nu)} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt,$$

where $s_0 := \min\{s > 0 \mid m(\tilde{t}(s)) = \nu\}, s_1 := \min\{s > 0 \mid \tilde{t}(s) = 0\}$.

By summing up the argument above, we have,

Lemma 3.4. *Let $\tilde{\gamma}(s) = (\tilde{t}(s), \tilde{\theta}(s))$ denote a geodesic emanating from the point $\tilde{p} := (\tilde{t}, \tilde{\theta})^{-1}(0, 0)$ with Clairaut constant $\nu \in (\inf m, m(0))$. Then $\tilde{\gamma}$ intersects $\tilde{t} = 0$ again at the point $(\tilde{t}, \tilde{\theta})^{-1}(0, \varphi(\nu))$. Here,*

$$\varphi(\nu) := 2\int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = 2\int_{0}^{\xi(\nu)} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt. \tag{3.8}$$

Lemma 3.5. *The length $l(\nu)$ of the subarc $(\tilde{t}(s), \tilde{\theta}(s)), 0 \le \tilde{\theta}(s) \le \varphi(\nu)$, of $\tilde{\gamma}(s)$ is given by*

$$l(\nu) = 2\int_{-\xi(\nu)}^{0} \frac{m}{\sqrt{m^2 - \nu^2}} dt = 2\int_{-\xi(\nu)}^{0} \frac{\sqrt{m^2 - \nu^2}}{m} dt + \nu\varphi(\nu), \tag{3.9}$$

and

$$\frac{\partial l}{\partial \nu}(\nu) = \nu\varphi'(\nu). \tag{3.10}$$

Proof. From (3.7), we obtain,

$$l(\nu) = 2\int_{-\xi(\nu)}^{0} \frac{m}{\sqrt{m^2-\nu^2}} dt.$$

Since

$$\frac{m}{\sqrt{m^2-\nu^2}} = \frac{\sqrt{m^2-\nu^2}}{m} + \frac{\nu^2}{m\sqrt{m^2-\nu^2}}$$

holds, we get

$$l(\nu) = 2\int_{-\xi(\nu)}^{0} \frac{\sqrt{m^2-\nu^2}}{m} dt + 2\int_{-\xi(\nu)}^{0} \frac{\nu^2}{m\sqrt{m^2-\nu^2}} dt.$$

Hence, by (3.8), we get (3.9). By differentiating $l(\nu)$ with respect to ν, we get,

$$l'(\nu) = 2\int_{-\xi(\nu)}^{0} \frac{\partial}{\partial \nu} \frac{\sqrt{m^2-\nu^2}}{m} dt + \varphi(\nu) + \nu\varphi'(\nu) = \nu\varphi'(\nu).$$

□

3.3 The decline of the function $\varphi(\nu)$

Let $\pi : \widetilde{M} = (R^1 \times R^1, d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2) \to M$ denote the universal covering space of M. We choose an arbitrary point $\tilde{p}$ of $\tilde{t}^{-1}(-\infty, 0]$, and we denote the cut locus of $\tilde{p}$ by $C_{\tilde{p}}$. Before proving some lemmas on the cut locus, let us review the structure of the cut locus of $\widetilde{M}$. We refer to [ShT] or [SST] on the structure of the cut locus of a 2-dimensional complete Riemannian manifold.

It is known that the cut locus has a local tree structure. Since $\widetilde{M}$ is simply connected, the cut locus has no circle. If two cut points x and y are in a common connected component of the cut locus, then x and y are connected by a unique rectifiable arc in the cut locus.

Since $\widetilde{M}$ is homeomorphic to R^2, we may define a global sector at each cut point. For general surfaces, only local sectors are defined (see [ShT], or [SST]). A global *sector* at each cut point x of the point $\tilde{p}$ is by definition a connected component of $\widetilde{M} \setminus \Gamma_x$, where Γ_x denotes the set of all points lying on a minimal geodesic segment joining $\tilde{p}$ to x. Let $c : [0, a] \to C_{\tilde{p}}$ denote a rectifiable arc in the cut locus. Then for each cut point $c(t), t \in (0, a)$, c bisects the sector at $c(t)$ containing $c[0, t)$ (respectively $c(t, a]$) . For each sector of the point $\tilde{p}$ on $\widetilde{M}$, there exists an end point of $C_{\tilde{p}}$, since $C_{\tilde{p}}$ has no circle. Here, a cut point q of $\tilde{p}$ is called an *end point* if q admits exactly one sector.

In this section, we assume that the Gaussian curvature G of M is increasing on the half meridian $t^{-1}(-\infty, 0] \cap \theta^{-1}(0)$ and that M has a reflective symmetry with respect to $t = 0$. Hence the Gaussian curvature of $\widetilde{M}$ is increasing on the lower half meridian $\tilde{t}^{-1}(-\infty, 0] \cap \tilde{\theta}^{-1}(0)$ and $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$.

Lemma 3.6. *Suppose that there exists a cut point of the point $\tilde{p}$ in $\tilde{t}^{-1}(-\infty, 0)$. Then there exist two minimal geodesic segments α and β joining $\tilde{p}$ to a cut point y of $\tilde{p}$ such that the global sector $D(\alpha, \beta)$ bounded by α and β has an end point of $C_{\tilde{p}}$ and $D(\alpha, \beta) \subset \tilde{t}^{-1}(-\infty, 0)$.*

Proof. Since the subset of cut points admitting at least two minimal geodesics is dense in the cut locus, the existence of two minimal geodesics α and β is clear (see [Bh]). Since $\widetilde{M}$ has a reflective symmetry with respective to $\tilde{t} = 0$, it is trivial that $D(\alpha, \beta) \subset \tilde{t}^{-1}(-\infty, 0)$. Let y denote the end point of α distinct from $\tilde{p}$. Since the proof is complete in the case where the cut point y is not an end point of the cut locus, we assume that y is an end point. Then, we get an arc c in the cut locus emanating from y. Any interior point y_1 on c is not an end point of the cut locus. It is clear that there exist two minimal geodesic segments joining $\tilde{p}$ and y_1 which bound a sector containing y as an end point of the cut locus. $\square$

Lemma 3.7. *For any unit speed minimal geodesic segment $\gamma : [0, L(\gamma)] \to \widetilde{M}$ joining $\tilde{p}$ to any end point x of $C_{\tilde{p}}$ in the domain $D(\alpha, \beta)$, x is conjugate to $\tilde{p}$ along γ and γ is shorter than α and β.*

Proof. Note that for any end point x of the cut locus, the set of all minimal geodesic segments joining $\tilde{p}$ to x is connected. Therefore, x is conjugate to $\tilde{p}$ along any minimal geodesic segments joining $\tilde{p}$ to the end point of the cut locus. Let $\gamma : [0, L(\gamma)] \to \widetilde{M}$ denote any minimal geodesic segment $\tilde{p}$ to an end point x of $C_{\tilde{p}} \cap D(\alpha, \beta)$. We will prove that γ is shorter than α and β. It follows from Theorem B in [ShT] or [IT] that there exists a unit speed arc $c : [0, l] \to C_{\tilde{p}}$ joining the end point x to y, where y denotes the end point of α distinct from $\tilde{p}$. Since the function $d(\tilde{p}, c(\tau))$ is a Lipschitz function, it follows from Lemma 7.29 in [WZ] that the function is differentiable for almost all τ and

$$d(\tilde{p}, c(l)) - d(\tilde{p}, y) = \int_0^l \frac{d}{d\tau} d(\tilde{p}, c(\tau)) d\tau \tag{3.11}$$

holds. From the Clairaut relation (3.3), the inner angle $\theta(\tau)$ at $c(\tau)$ of the sector containing $c[0, \tau)$ is less than π. Hence, by the first variation formula, we get

$$\frac{d}{d\tau} d(\tilde{p}, c(\tau)) = \cos \frac{\theta(\tau)}{2} > 0$$

for almost all τ. Notice that for each $\tau \in (0, l)$, the curve c bisects the sector at $c(\tau)$ containing $c[0, \tau)$. Therefore, from (3.11),

$$L(\alpha) = L(\beta) = d(\tilde{p}, c(l)) > d(\tilde{p}, y) = L(\gamma).$$

$\square$

Lemma 3.8. *Let q be a point on $\tilde{\theta}^{-1}(0)$ and u_0 any real number. Then $d(q, c(\theta))$ is strictly increasing on $[0, \infty)$. Here $c : [0, \infty) \to \widetilde{M}$ denotes $c(\theta) = (u_0, \theta)$ in the coordinates $(\tilde{t}, \tilde{\theta})$ and $d(\cdot, \cdot)$ denotes the Riemannian distance function on $\widetilde{M}$.*

Proof. Choose any positive numbers $\theta_1 < \theta_2$. Let $\alpha_i, i = 1, 2$, denote minimal geodesic segments joining the point q to $c(\theta_i)$ respectively. Since $\theta_2 > \theta_1$, there exists an intersection $\alpha_2(t_2)$ of α_2 and the meridian $\tilde{\theta} = \theta_1$. The point $c(\theta_1)$ is the unique nearest point on $\tilde{t} = u_0$ from $\alpha_2(t_2)$. Hence,

$$d(\alpha_2(t_2), c(\theta_1)) < d(\alpha_2(t_2), c(\theta_2)).$$

Therefore, by the triangle inequality, we get

$$d(q, c(\theta_2)) = d(q, \alpha_2(t_2)) + d(\alpha_2(t_2), c(\theta_2)) > d(q, \alpha_2(t_2)) + d(\alpha_2(t_2), c(\theta_1)) \geq d(q, c(\theta_1)).$$

This implies that $d(q, c(\theta))$ is strictly increasing on $[0, \infty)$. $\square$

Lemma 3.9. *Suppose that $\gamma : [0, L(\gamma)] \to \widetilde{M}$ is a minimal geodesic segment joining $\tilde{p}$ to an end point $x \in C_{\tilde{p}}$, which is a point in the sector $D(\alpha, \beta)$ bounded by two minimal geodesic segments α and β emanating from $\tilde{p}$. Then, for any $s \in [0, L(\gamma)]$, $\tilde{t}(\alpha(s)) \geq \tilde{t}(\gamma(s)) \geq \tilde{t}(\beta(s))$ holds. Here we assume that*

$$\angle(\alpha'(0), (\partial/\partial\tilde{t})_{\tilde{p}}) < \angle(\gamma'(0), (\partial/\partial\tilde{t})_{\tilde{p}}) < \angle(\beta'(0), (\partial/\partial\tilde{t})_{\tilde{p}}),$$

where $\angle(\cdot, \cdot)$ denotes the angle made by two tangent vectors.

Proof. From (3.5), it follows that for sufficiently small $s > 0$, $\tilde{t}(\alpha(s)) > \tilde{t}(\gamma(s)) > \tilde{t}(\beta(s))$ holds. Hence the set $A := \{s \in (0, L(\gamma)) \mid \tilde{t}(\alpha(s)) > \tilde{t}(\gamma(s)) > \tilde{t}(\beta(s))\}$ is a nonempty open subset of $(0, L(\gamma))$. Let $(0, s_0)$ denote the connected component of A. It is sufficient to prove that $s_0 = L(\gamma)$. Suppose that $s_0 < L(\gamma)$. Thus, $\tilde{t}(\alpha(s_0)) = \tilde{t}(\gamma(s_0))$ or $\tilde{t}(\gamma(s_0)) = \tilde{t}(\beta(s_0))$ holds, since A is open. By applying Lemma 3.8 for $u_0 := \tilde{t}(\alpha(s_0))$ and $\tilde{t}(\beta(s_0))$, we get $\alpha(s_0) = \gamma(s_0)$ or $\gamma(s_0) = \beta(s_0)$, which is a contradiction. $\square$

Lemma 3.10. *For any point $\tilde{p} \in \tilde{t}^{-1}(-\infty, 0]$, there does not exist a cut point of $\tilde{p}$ in $\tilde{t}^{-1}(-\infty, 0)$. In particular, the cut locus of $\tilde{p}$ is a subset of $\tilde{t}^{-1}(0)$ if $\tilde{t}(\tilde{p}) = 0$. This* *implies that the cut locus C_p of a point $p \in t^{-1}(0)$ is a subset of $\theta^{-1}(\pi) \cup t^{-1}(0)$. Here the coordinates (t, θ) are chosen so as to satisfy $\theta(p) = 0$.*

Proof. Suppose that there exist a cut point of $\tilde{p}$ in $\tilde{t}^{-1}(-\infty, 0)$. By Lemma 3.6, there exist two minimal geodesic segments α and β joining a cut point y of $\tilde{p}$ which bound a sector $D(\alpha, \beta)$ containing an end point x of $C_{\tilde{p}}$. Let $\gamma : [0, L(\gamma)] \to \widetilde{M}$ be a unit speed geodesic segment joining $\tilde{p}$ to the end point x. From Lemmas 3.6 and 3.9, it follows that for any $s \in [0, L(\gamma)]$,

$$0 \geq \tilde{t}(\alpha(s)) \geq \tilde{t}(\gamma(s)) \geq \tilde{t}(\beta(s))$$

holds. Since the Gaussian curvature G is increasing on each lower half meridian, we obtain

$$G(\alpha(s)) \geq G(\gamma(s)) \geq G(\beta(s)).$$

By applying the Rauch comparison theorem for the pair of geodesic segments $\alpha|_{[0,L(\gamma)]}$ and γ, $\tilde{p}$ admits a conjugate point on $\alpha|_{[0,L(\gamma)]}$ along α.

This contradicts the fact that α is minimal. Since $\widetilde{M}$ is symmetric with respect to $\tilde{t} = 0$, the cut locus of $\tilde{p}$ is a subset of $\tilde{t}^{-1}(0)$, if $\tilde{t}(\tilde{p}) = 0$. This implies that $C_p \subset \theta^{-1}(\pi) \cup t^{-1}(0)$ for the point $p = t^{-1}(0) \cap \theta^{-1}(0)$. □

Proposition 3.11. *Let M be a complete Riemannian manifold $R^1 \times S^1$ with a warped product metric $ds^2 = dt^2 + m(t)^2 d\theta^2$ of the real line (R^1, dt^2) and the unit circle $(S^1, d\theta^2)$. Here the warping function $m : R \to (0, \infty)$ is a smooth even function. If the Gaussian curvature is positive on the equator and decreasing on the upper half meridian $t^{-1}(0, \infty) \cap \theta^{-1}(0)$, then the function $\varphi(\nu)$ is decreasing on $(\inf m, m(0))$.*

Proof. Let $\widetilde{M} := (R^1 \times R^1, d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2)$ denote the universal covering space of M. Choose any point $\tilde{p}$ on $\tilde{t}^{-1}(0)$. For each $\nu \in (\inf m, m(0))$, let $\alpha_\nu : [0, \infty) \to \widetilde{M}$ denote the geodesic emanating from the point $\tilde{p} = \alpha_\nu(0)$ with Clairaut constant ν and with $(\tilde{t} \circ \alpha_\nu)'(0) < 0$. From the Clairaut relation, we get $\angle((\partial/\partial\tilde{\theta})_{\tilde{p}}, \alpha_\nu'(0)) = \cos^{-1} \nu/m(0)$. Choose any $\nu_1 < \nu_2$ with $\nu_1, \nu_2 \in (\inf m, m(0))$. Since

$$\cos^{-1} \frac{\nu_2}{m(0)} < \cos^{-1} \frac{\nu_1}{m(0)},$$

it follows from Lemma 3.10 that α_{ν_1} does not cross the domain bounded by the subarc of α_{ν_2} and $\tilde{t}^{-1}(0) \cap \tilde{\theta}^{-1}[\tilde{\theta}(\tilde{p}), \tilde{\theta}(\tilde{p}) + \varphi(\nu_2)]$. This implies that $\varphi(\nu_1) \geq \varphi(\nu_2)$. Therefore, $\varphi(\nu)$ is decreasing on $(\inf m, m(0))$. □

3.4 The cut locus of a point on $\widetilde{M}$

Choose any point q on $\widetilde{M}$ with $-t_0 < \tilde{t}(q) < 0$, where $t_0 := \sup\{\, t > 0 \mid m'(t) < 0\}$. Without loss of generality, we may assume that $\tilde{\theta}(q) = 0$. We consider two geodesics α_ν and β_ν emanating from the point $q = \alpha_\nu(0) = \beta_\nu(0)$ with Clairaut constant $\nu > 0$. Here we assume that

$$\angle((\partial/\partial\tilde{t})_q, \alpha_\nu'(0)) > \angle((\partial/\partial\tilde{t})_q, \beta_\nu'(0)).$$

Lemma 3.12. *The two geodesics α_ν and β_ν intersect again at the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$ if $\nu \in (\inf m, m(0))$, where $u := -\tilde{t}(q)$.*

Proof. Suppose that $\nu \in (\inf m, m(0))$. Since α_ν is tangent to the parallel arc $\tilde{t} = -\xi(\nu)$, it follows from (3.6) that

$$\tilde{\theta}(\alpha_\nu(s_1)) - \tilde{\theta}(\alpha_\nu(0)) = \int_{-\xi(\nu)}^{-u} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt,$$

where $s_1 := \min\{s > 0 \mid \tilde{t}(\alpha_\nu(s)) = -\xi(\nu)\}$, and

$$\tilde{\theta}(\alpha_\nu(s_2)) - \tilde{\theta}(\alpha_\nu(s_1)) = \int_{-\xi(\nu)}^{u} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt,$$

where $s_2 := \min\{s > 0 \mid \tilde{t}(\alpha_\nu(s)) = u\}$. Hence, we obtain,

$$\tilde{\theta}(\alpha_\nu(s_2)) - \tilde{\theta}(\alpha_\nu(0)) = \int_{-\xi(\nu)}^{u} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt + \int_{-\xi(\nu)}^{-u} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt. \tag{3.12}$$

Since m is an even function,

$$\int_{-\xi(\nu)}^{u} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = \int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt + \int_{-u}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt$$

holds. Therefore, by (3.12),

$$\tilde{\theta}(\alpha_\nu(s_2)) - \tilde{\theta}(\alpha_\nu(0)) = 2\int_{-\xi(\nu)}^{0} \frac{\nu}{\sqrt{m^2 - \nu^2}} dt = \varphi(\nu).$$

This implies that α_ν passes through the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$. On the other hand, after β_ν is tangent to $\tilde{t} = \xi(\nu)$ at $\beta_\nu(s_1^+)$, where $s_1^+ := \min\{s > 0 \mid \tilde{t}(\beta_\nu(s)) = \xi(\nu)\}$, the geodesic intersects $\tilde{t} = u$ again at $\beta_\nu(s_2^+)$, where $s_2^+ := \min\{s > s_1^+ \mid \tilde{t}(\beta_\nu(s)) = u\}$. By the similar computation as above, we get

$$\tilde{\theta}(\beta_\nu(s_2^+)) - \tilde{\theta}(\beta_\nu(0)) = \varphi(\nu).$$

This implies that α_ν and β_ν pass through the common point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$. $\square$

Lemma 3.13. *The two geodesic segments $\alpha_\nu|_{[0,s_2]}$ and $\beta_\nu|_{[0,s_2^+]}$ have the same length and its length equals $l(\nu)$, which is defined in Lemma 3.5. In particular, $s_2 = s_2^+$. Here, s_2 and s_2^+ denote the numbers defined in the proof of Lemma 3.12.*

Proof. From (3.7), we have

$$L(\alpha_\nu|_{[0,s_1]}) = \int_{-\xi(\nu)}^{-u} \frac{m}{\sqrt{m^2-\nu^2}}dt, \tag{3.13}$$

and

$$L(\alpha_\nu|_{[s_1,s_2]}) = \int_{-\xi(\nu)}^{u} \frac{m}{\sqrt{m^2-\nu^2}}dt = \int_{-\xi(\nu)}^{0} \frac{m}{\sqrt{m^2-\nu^2}}dt + \int_{0}^{u} \frac{m}{\sqrt{m^2-\nu^2}}dt,$$

where s_1 denotes the number defined in the proof of Lemma 3.12. Since m is even

$$L(\alpha_\nu|_{[s_1,s_2]}) = \int_{-\xi(\nu)}^{0} \frac{m}{\sqrt{m^2-\nu^2}}dt + \int_{-u}^{0} \frac{m}{\sqrt{m^2-\nu^2}}dt. \tag{3.14}$$

Therefore, we get, by (3.9), (3.13) and (3.14),

$$L(\alpha_\nu|_{[0,s_2]}) = 2\int_{-\xi(\nu)}^{0} \frac{m}{\sqrt{m^2-\nu^2}}dt = l(\nu).$$

Analogously we have,

$$L(\beta_\nu|_{[0,s_2^+]}) = l(\nu).$$

□

Lemma 3.14. *Let q be a point on $\widetilde{M}$ with $|\tilde{t}(q)| \in (0, t_0)$. For any $\nu \in (\inf m, m(u)]$, where $u = -\tilde{t}(q)$, $\alpha_\nu|_{[0,s_2(\nu)]}$ and $\beta_\nu|_{[0,s_2(\nu)]}$ are minimal geodesic segments joining q to the point $(\tilde{t},\tilde{\theta})^{-1}(u, \tilde{\theta}(q) + \varphi(\nu))$, and in particular, $\{(\tilde{t},\tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq \varphi(m(u)) + \tilde{\theta}(q)\}$ is a subset of the cut locus of the point q. Here, $s_2(\nu) := \min\{s > 0 \mid \tilde{t}(\alpha_\nu(s)) = u\}$ for each $\nu \in (\inf m, m(0))$.*

Proof. Without loss of generality, we may assume that $\tilde{\theta}(q) = 0$. We will prove that $\alpha_\nu|_{[0,s_2(\nu)]}$ is a minimal geodesic segment joining q to the point $\alpha_\nu(s_2(\nu)) = (\tilde{t},\tilde{\theta})^{-1}$ $(u, \varphi(\nu))$. Suppose that $\alpha_{\nu_0}|_{[0,s_2(\nu_0)]}$ is not minimal for some $\nu_0 \in (\inf m, m(u)]$. Here we assume that ν_0 is the minimum solution $\nu = \nu_0$ of $\varphi(\nu) = \varphi(\nu_0)$.

Let $\alpha : [0, d(q,x)] \to M$ be a minimal geodesic segment joining q to $x := \alpha_{\nu_0}(s_2(\nu_0)) = (\tilde{t},\tilde{\theta})^{-1}(u, \varphi(\nu_0))$. Hence, $\varphi(\nu_1) = \varphi(\nu_0) = \tilde{\theta}(x)$ and α equals $\alpha_{\nu_1}|_{[0,s_2(\nu_1)]}$ or $\beta_{\nu_1}|_{[0,s_2(\nu_1)]}$, where $\nu_1 \in (\inf m, m(0))$ denotes the Clairaut constant of α. By Proposition 3.11, $\varphi(\nu) =$

$\varphi(\nu_0)$ for any $\nu \in [\nu_0, \nu_1]$. Hence, by Lemmas 3.5 and 3.13 we get,

$$s_2(\nu_1) = L(\alpha) = L(\alpha_{\nu_1}|_{[0,s_2(\nu_1)]}) = L(\alpha_{\nu_0}|_{[0,s_2(\nu_0)]}) = s_2(\nu_0).$$

This implies that $\alpha_{\nu_0}|_{[0,s_2(\nu_0)]}$ is minimal, which is a contradiction, since we assumed that $\alpha_{\nu_0}|_{[0,s_2(\nu_0)]}$ is not minimal. Therefore, by Lemma 3.13, for any $\nu \in (\inf m, m(u)]$, the geodesic segments $\alpha_\nu|_{[0,s_2(\nu)]}$ and $\beta_\nu|_{[0,s_2(\nu)]}$ are minimal geodesic segments joining q to the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu)) = \alpha_\nu(s_2(\nu))$. In particular, the point $\alpha_\nu(s_2(\nu)) = \beta_\nu(s_2(\nu))$ is a cut point of q. $\square$

Proposition 3.15. *The cut locus of the point q in Lemma 3.14 equals the set*

$$\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}.$$

Here the coordinates $(\tilde{t}, \tilde{\theta})$ are chosen so as to satisfy $\tilde{\theta}(q) = 0$.

Proof. By Lemma 3.14, geodesic segments $\alpha_\nu|_{[0,s_2(\nu)]}$ and $\beta_\nu|_{[0,s_2(\nu)]}$ are minimal geodesic segments for any $\nu \in (\inf m, m(u)]$. Hence their limit geodesics $\alpha^- := \alpha_{\inf m}$ and $\beta^+ := \beta_{\inf m}$ are rays, that is, any their subarcs are minimal.

Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{\theta} = 0$, it is trivial from Lemma 3.14 that the set $\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}$ is a subset of the cut locus of q. Suppose that there exists a cut point $y \notin \{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}$. Without loss of generality, we may assume that $\tilde{\theta}(y) > 0 = \tilde{\theta}(q)$ and $\tilde{t}(q) = -u < 0$. From Lemma 3.10, $\tilde{t}(y) > 0$ and y is not a point in the unbounded domain cut off by two rays α^- and β^+, and hence the point lies in the domain D^+ cut off by β^+ and the submeridian $\tilde{t} > -u, \tilde{\theta} = \tilde{\theta}(q) = 0$. Since the cut locus of C_q has a tree structure, there exists an end point x of the cut locus in the D^+. Hence, x is conjugate to q for any minimal geodesic segment γ joining q to x. Since such a minimal geodesic γ runs in the domain D^+, the Clairaut constant of the segment is positive and less than $\inf m$. From the Clairaut relation (3.3), any geodesic cannot be tangent to any parallel arc $\tilde{t} = c$, if the Clairaut constant is positive and less than $\inf m$. From Corollary 7.2.1 in [SST], γ has no conjugate point of q, which is a contradiction. $\square$

Lemma 3.16. *Let q be a point on $\widetilde{M}$ with $|\tilde{t}(q)| \geq t_0$. Then the cut locus of q is empty.*

Proof. Suppose that the cut locus of a point q with $|\tilde{t}(q)| \geq t_0$ is nonempty. Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$, we may assume that $\tilde{t}(q) \leq -t_0$. Hence by Lemma 3.10, there exists an end point x of the cut locus C_q in $\tilde{t}^{-1}(0, \infty)$. Let $\gamma : [0, d(q, x)] \to \widetilde{M}$ denote a minimal geodesic segment joining q to x. Then x is conjugate to q along γ, since x is an end point of C_q. Since $\tilde{\theta}(x) > 0 = \tilde{\theta}(q)$, the Clairaut constant ν of γ is positive, by (3.2). Moreover, from the Clairaut relation (3.3), the Clairaut constant ν is less than $\inf m = m(t_0)$, since γ intersects $\tilde{t} = -t_0$. Therefore, γ cannot be tangent to any parallel arc $\tilde{t} = c$. From Corollary 7.2.1 in [SST], γ has no conjugate point of q, which is a contradiction. $\square$

Now our Main theorem is clear from Proposition 3.15 and Lemma 3.16.

Chapter 4

The case where the half period function is decreasing for a cylinder of revolution

The following structure theorem was proved in [C1] for a class of surfaces of revolution homeomorphic to a cylinder.

Theorem Let (M, ds^2) be a complete Riemannian manifold $R^1 \times S^1$ with a warped product metric $ds^2 = dt^2 + m(t)^2 d\theta^2$ of the real line (R^1, dt^2) and the unit circle $(S^1, d\theta^2)$. Suppose that the warping function m is a positive-valued even function and the Gaussian curvature of M is decreasing along the half meridian $t^{-1}[0,\infty] \cap \theta^{-1}(0)$. If the Gaussian curvature of M is positive on $t = 0$, then the structure of the cut locus C_q of a point $q \in \theta^{-1}(0)$ in M is given as follows:

1. The cut locus C_q is the union of a subarc of the parallel $t = -t(q)$ opposite to q and the meridian opposite to q if $|t(q)| < t_0 := \sup\{t > 0 | m'(t) < 0\}$ and $\varphi(m(t(q))) < \pi$. More precisely,

$$C_q = \theta^{-1}(\pi) \cup (t^{-1}(-t(q)) \cap \theta^{-1}[\varphi(m(t(q))), 2\pi - \varphi(m(t(q)))]).$$

2. The cut locus C_q is the meridian $\theta^{-1}(\pi)$ opposite to q if $\varphi(m(t(q))) \geq \pi$ or if $|t(q)| \geq t_0$.

Here, the *half period function* $\varphi(\nu)$ on $(\inf m, m(0))$ is defined as

$$\varphi(\nu) := 2\int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = 2\int_{0}^{\xi(\nu)} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt, \tag{4.1}$$

where $\xi(\nu) := \min\{t > 0 | m(t) = \nu\}$. Notice that the point q is an arbitrarily given point if the coordinates (t, θ) are chosen so as to satisfy $\theta(q) = 0$.

Crucial properties of the manifold (M, ds^2) in the theorem above are

1. M has a reflective symmetry with respect to a parallel.
2. The Gaussian curvature is decreasing along each upper half meridian.

In this chapter, the second property is replaced by the following property:

The cut locus of a point on $\tilde{t} = 0$ is a nonempty subset of $\tilde{t} = 0$, for the universal covering space $(\tilde{M}, d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2)$ of a cylinder of revolution $(M, dt^2 + m(t)^2 d\theta^2)$ with a reflective symmetry with respect to the parallel $t = 0$.

We will prove the following structure theorem of the cut locus for a cylinder of revolution satisfying the property above.

Main Theorem. *Let (M, ds^2) denote a complete Riemannian manifold $R^1 \times S^1$ with a warped product metric $ds^2 = dt^2 + m(t)^2 d\theta^2$ of the real line (R^1, dt^2) and the unit circle $(S^1, d\theta^2)$, and by $(\widetilde{M}, d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2)$ we denote the universal covering space of (M, ds^2). Suppose that m is an even positive-valued function. If the cut locus of a point on $\tilde{t}^{-1}(0)$ is a nonempty subset of $\tilde{t}^{-1}(0)$, then the cut locus C_q of a point q of M with $|t(q)| < t_0 := \sup\{t > 0 \mid m'|_{(0,t)} < 0\}$ equals the union of a subarc of the parallel $t = -t(q)$ opposite to q and the meridian opposite to q. More precisely, there exists a number $t_\pi \in [0, t_0)$ such that for any point q with $|t(q)| < t_\pi$,*

$$C_q = \theta^{-1}(\pi) \cup \big(t^{-1}(-t(q)) \cap \theta^{-1}[\varphi(m(t(q))), 2\pi - \varphi(m(t(q)))]\big)$$

and for any point q with $t_\pi \leq |t(q)| < t_0$, $C_q = \theta^{-1}(\pi)$. Moreover, if t_0 is finite, then $C_q = \theta^{-1}(\pi)$ for any point q with $|t(q)| = t_0$.

Here the coordinates (t, θ) are chosen so as to satisfy $\theta(q) = 0$. Notice that the domain of the half period function $\varphi(\nu)$ is $(m(t_0), m(0))$ (respectively $(\inf m, m(0))$) if t_0 is finite (respectively infinite).

Remark 4.1. If the Gaussian curvature of the manifold M in the Main Theorem is nonpositive on $\tilde{t}^{-1}(t_0, \infty)$, then the cut locus C_q of any point q with $|t(q)| > t_0$ is equal to $\theta^{-1}(\pi)$, the meridian opposite to q.

We refer to [C1], [SST] and [ST] for some fundamental properties of geodesics on a surface of revolution and the structure theorem of the cut locus on a surface.

4.1 A necessary and sufficient condition for $\varphi(\nu)$ to be decreasing

A complete Riemannian manifold (M, ds^2) homeomorphic to $R^1 \times S^1$ is called a *cylinder of revolution* if $ds^2 = dt^2 + m(t)^2 d\theta^2$ is a warped product metric of the real line (R^1, dt^2) and the unit circle $(S^1, d\theta^2)$.

Throughout this paper, we assume that the warping function m of a cylinder of revolution M is an even function. Hence M has a reflective symmetry with respect to $t = 0$, which is called the *equator*. Let $(\widetilde{M}, d\tilde{s}^2)$ denote the universal covering space of (M, ds^2). Thus $d\tilde{s}^2 = d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2$. Since $m'(0) = 0$, it follows from Lemma 7.1.4 in [SST] that the equator $t = 0$ and $\tilde{t} = 0$ are geodesics in M and $\widetilde{M}$ respectively.

The following lemma is a corresponding one to Lemma 3.2 in [BCST] in the case of a two-sphere of revolution.

Lemma 4.2. *If the cut locus of a point in $\tilde{t}^{-1}(0)$ is a nonempty subset of $\tilde{t}^{-1}(0)$, then the Gaussian curvature of $\widetilde{M}$ is positive on $\tilde{t}^{-1}(0)$ and for any $t > 0$ satisfying $m'|_{(0,t)} < 0$, the function $\varphi(\nu)$ is decreasing on $(m(t), m(0))$.*

Proof. Let q be an end point of the cut locus of a point $p \in \tilde{t}^{-1}(0)$. Since the end point q is conjugate to p along the subarc of $\tilde{t}^{-1}(0)$, the Gaussian curvature on $\tilde{t}^{-1}(0)$ is positive. We omit the proof of the second claim, since the proof of Proposition 4.6 in [C1] is applicable. □

Lemma 4.3. *Suppose that the Gaussian curvature of $\widetilde{M}$ is positive on $\tilde{t}^{-1}(0)$. Let $t > 0$ be any number satisfying $m'|_{(0,t)} < 0$. If $\varphi(\nu)$ is decreasing on $(m(t), m(0))$ then for any point $\tilde{p} \in \tilde{t}^{-1}(0)$, $C_{\tilde{p}} \cap \tilde{t}^{-1}(-t, t)$ is a nonempty subset of $\tilde{t}^{-1}(0)$. Here $C_{\tilde{p}}$ denotes the cut locus of $\tilde{p}$.*

Proof. Choose an arbitrary point $\tilde{p} \in \tilde{t}^{-1}(0)$ and fix it. Since the Gaussian curvature is positive constant on $\tilde{t} = 0$, there exists a conjugate point of $\tilde{p}$ along the subarc of $\tilde{t} = 0$. Thus, $C_{\tilde{p}} \cap \tilde{t}^{-1}(-t, t)$ is nonempty. We omit the proof of the claim that $C_{\tilde{p}} \cap \tilde{t}^{-1}(-t, t)$ is a subset of $\tilde{t}^{-1}(0)$, since the proof of Lemma 3.3 in [BCST] is still valid in our case. □

Combining Lemmas 4.2 and 4.3 we get

Proposition 4.4. *Suppose that $m' \neq 0$ on $(0, \infty)$. Then the cut locus of a point on $\tilde{t}^{-1}(0)$ is a nonempty subset of $\tilde{t}^{-1}(0)$ if and only if the Gaussian curvature of $\widetilde{M}$ is positive on $\tilde{t}^{-1}(0)$ and the half period function $\varphi(\nu)$ defined by (4.1) is decreasing on $(\inf m, m(0))$.*

4.2 Preliminaries

From now on, we assume that *the cut locus of a point on $\tilde{t} = 0$ is a nonempty subset of $\tilde{t} = 0$*. Hence, from Lemma 4.2, the function $\varphi(\nu)$ is decreasing on $(m(t_0), m(0))$, where $t_0 := \sup\{\, t > 0 \mid m'|_{(0,t)} < 0\}$ and $m(t_0)$ means $\inf m$ when $t_0 = \infty$. For each $\nu \in [0, m(0))$ let $\gamma_\nu : [0, \infty) \to \widetilde{M}$ denote a unit speed geodesic emanating from the point $\tilde{p} := (\tilde{t}, \tilde{\theta})^{-1}(0, 0)$ on $\tilde{t}^{-1}(0)$ with Clairaut constant ν. It is known (see [C1], for example) that γ_ν intersects $\tilde{t}^{-1}(0)$ again at the point $(\tilde{t}, \tilde{\theta})^{-1}(0, \varphi(\nu))$ if ν is greater than $m(t_0)$. Notice that γ_ν is a submeridian of $\tilde{\theta} = 0$, when $\nu = 0$.

Lemma 4.5. *If $0 \leq \nu \leq m(t_0)$, then γ_ν is not tangent to any parallel arc $\tilde{t} = c$. In particular, the geodesic does not intersect $\tilde{t} = 0$ again.*

Proof. Since there does not exist a cut point of $\tilde{p}$ in $\tilde{t} \neq 0$, the subarc $\gamma_\nu|_{[0,l(\nu)]}$ of γ_ν is minimal for each $\nu \in (m(t_0), m(0))$. Here $l(\nu)$ denotes the length of the subarc of γ_ν having end points $\tilde{p}$ and $(\tilde{t}, \tilde{\theta})^{-1}(0, \varphi(\nu))$. Therefore, the limit geodesic $\gamma_{m(t_0)} = \lim_{\nu \searrow m(t_0)} \gamma_\nu|_{[0,l(\nu)]}$ is a ray emanating from $\tilde{p}$ and in particular, $\gamma_{m(t_0)}$ is not tangent to any parallel arc and does not intersect $\tilde{t} = 0$ again. We will prove that for any $\nu \in [0, m(t_0))$, γ_ν is not tangent to any parallel arc. Suppose that for some $\nu_0 \in (0, m(t_0))$, γ_{ν_0} is tangent to a parallel arc. Since $\widetilde{M}$ has a reflection symmetry with respect to $\tilde{t} = 0$, we may assume that $(\tilde{t} \circ \gamma_{\nu_0})'(0) < 0$ and $(\tilde{t} \circ \gamma_{m(t_0)})'(0) < 0$. By applying the Clairaut relation at the point $\tilde{p}$, $\gamma_{\nu_0}|_{(0,t)}$ lies in the domain D cut off by $\gamma_{m(t_0)}$ and the submeridian γ_0 of $\tilde{\theta} = 0$ for some positive t. Since there does not exist a cut point of $\tilde{p}$ in $\tilde{t}^{-1}(-\infty, 0)$, the geodesic γ_{ν_0} does not intersect $\gamma_{m(t_0)}$ again. Hence $\gamma_{\nu_0}|_{(0,\infty)}$ lies in the domain D. Since γ_{ν_0} is tangent to a parallel arc, the geodesic intersects $\tilde{t} = 0$ again, which is a contradiction. $\square$

Lemma 4.6. *Let $\tilde{\gamma}_\nu : R \to \widetilde{M}$ denote a unit speed geodesic with Clairaut constant $\nu \in (0, m(t_0)]$. If $\tilde{\gamma}_\nu$ passes through a point of $\tilde{t}^{-1}(-t_0, t_0)$, then $\tilde{\gamma}_\nu$ is not tangent to any parallel arc $\tilde{t} = c$.*

Proof. First, we will prove that $\tilde{\gamma}_\nu$ intersects $\tilde{t} = 0$ for any $\nu \in [0, m(t_0)]$. Supposing that $\tilde{\gamma}_\nu$ does not intersect $\tilde{t} = 0$ for some $\nu \in [0, m(t_0)]$, we will get a contradiction. Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$, we may assume that $(\tilde{t} \circ \tilde{\gamma}_\nu)(s) < 0$ for any real number s. By the Clairaut relation, $(\tilde{t} \circ \tilde{\gamma}_\nu)'(s) \neq 0$ for any s satisfying $-t_0 < \tilde{t} \circ \tilde{\gamma}_\nu(s) < 0 < t_0$. From the assumptions, we may assume that $\tilde{t} \circ \tilde{\gamma}_\nu(0) \in (-t_0, 0)$. If $(\tilde{t} \circ \tilde{\gamma}_\nu)'(0) > 0$ (respectively $(\tilde{t} \circ \tilde{\gamma}_\nu)'(0) < 0$) then $\tilde{t} \circ \tilde{\gamma}_\nu(s)$ is increasing (respectively decreasing) and bounded above by 0. Thus, there exists a unique limit $-t_0 < \tilde{t}_1 := \lim_{s \to \infty} \tilde{t} \circ \tilde{\gamma}_\nu(s) \leq 0$ (respectively $-t_0 < \tilde{t}_1 := \lim_{s \to -\infty} \tilde{t} \circ \tilde{\gamma}_\nu(s) \leq 0$). It follows from Lemma 7.1.7 in [SST] that $m'(\tilde{t}_1) = 0$ and $m(\tilde{t}_1) = \nu$. This is a contradiction, since

$\nu \in [0, m(t_0)]$ and $-t_0 < \tilde{t}_1 \leq 0$. Therefore, $\tilde{\gamma}_\nu$ intersects $\tilde{t} = 0$ for any $\nu \in [0, m(t_0)]$, and hence by Lemma 4.5, the geodesic is not tangent to any parallel arc. $\square$

Lemma 4.7. *If $t_0 = \sup\{\, t > 0 \mid m'|_{(0,t)} < 0\}$ is finite, then any subarc of the parallel arc $\tilde{t} = -t_0$ is minimal, i.e., the parallel arc is a straight line. Hence, $\tilde{t} = t_0$ is also a straight line.*

Proof. Since $m'(t_0) = 0$, the parallel arc $\tilde{t} = -t_0$ is a geodesic by Lemma 7.1.4 in [SST]. Let c be a geodesic emanating from a point on $\tilde{t} = -t_0$ which is not tangent to $\tilde{t} = -t_0$. By Lemma 4.6, c is not tangent to any parallel arc. In particular, c does not intersect $\tilde{t} = -t_0$ again. This implies that $\tilde{t} = -t_0$ is a straight line. Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$, $\tilde{t} = t_0$ is also a straight line. $\square$

4.3 The cut locus of a point in $\widetilde{M}$

Choose any point q in $\widetilde{M}$ with $-t_0 < \tilde{t}(q) < 0$. Without loss of generality, we may assume that $\tilde{\theta}(q) = 0$. For each $\nu \in [0, m(0))$ let $\gamma_\nu : [0, \infty) \to \widetilde{M}$ denote a geodesic emanating from the point $\tilde{p} := (\tilde{t}, \tilde{\theta})^{-1}(0, 0)$ on $\tilde{t}^{-1}(0)$ with Clairaut constant ν. The geodesic γ_ν intersects $\tilde{t} = 0$ again at the point $(\tilde{t}, \tilde{\theta})^{-1}(0, \varphi(\nu))$, if $\nu > m(t_0)$.

We consider two geodesics α_ν and β_ν emanating from the point $q = \alpha_\nu(0) = \beta_\nu(0)$ with Clairaut constant $\nu > 0$. Here we assume that the angle $\angle((\partial/\partial\tilde{t})_q, \alpha_\nu'(0))$ made by the tangent vectors $(\partial/\partial\tilde{t})_q$ and $\alpha'_{\nu(0)}$ is greater than the angle $\angle((\partial/\partial\tilde{t})_q, \beta_\nu'(0))$ by $(\partial/\partial\tilde{t})_q$ and $\beta_\nu'(0)$, if $\nu < m(t(q))$. Notice that $\alpha_\nu = \beta_\nu$ if $\nu = m(t(q))$.

It follows from Lemma 5.1 in [C1] that the geodesics α_ν and β_ν intersect again at the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$, where $u := -\tilde{t}(q)$, if $\nu \in (m(t_0), m(u))$. The subarcs of α_ν and β_ν having end points q and $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$ have the same length and its length equals $l(\nu)$, where $l(\nu)$ denotes the length the subarc of γ_ν having end points $\tilde{p}$ and $(\tilde{t}, \tilde{\theta})^{-1}(0, \varphi(\nu))$.

Lemma 4.8. *Let q be a point in $\widetilde{M}$ with $|\tilde{t}(q)| \in (0, t_0)$. Then, for any $\nu \in (m(t_0), m(u)]$, where $u = -\tilde{t}(q)$, $\alpha_\nu|_{[0,l(\nu)]}$ and $\beta_\nu|_{[0,l(\nu)]}$ are minimal geodesic segments joining q to the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \tilde{\theta}(q) + \varphi(\nu))$, and in particular, $\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq \varphi(m(u)) + \tilde{\theta}(q)\}$ is a subset of the cut locus of the point q.*

Proof. Without loss of generality, we may assume that $\tilde{\theta}(q) = 0$ and $-t_0 < \tilde{t}(q) < 0$. We will prove that $\alpha_\nu|_{[0,l(\nu)]}$ is a minimal geodesic segment joining q to the point $\alpha_\nu(l(\nu)) = (\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$. Suppose that $\alpha_{\nu_0}|_{[0,l(\nu_0)]}$ is not minimal for some $\nu_0 \in (m(t_0), m(u)]$. Here we assume that ν_0 is the minimum solution $\nu = \nu_0$ of $\varphi(\nu) = \varphi(\nu_0)$.

Let $\alpha : [0, d(q,x)] \to M$ be a unit speed minimal geodesic segment joining q to $x := \alpha_{\nu_0}(l(\nu_0)) = (\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu_0))$. Hence, $\varphi(\nu_1) = \varphi(\nu_0) = \tilde{\theta}(x)$ and α equals $\alpha_{\nu_1}|_{[0,l(\nu_1)]}$ or $\beta_{\nu_1}|_{[0,l(\nu_1)]}$, where $\nu_1 \in (m(t_0), m(u))$ denotes the Clairaut constant of α. By Lemma 4.2, $\varphi(\nu) = \varphi(\nu_0)$ for any $\nu \in [\nu_0, \nu_1]$. Hence, by Lemma 3.2 in [C1] we get, $l(\nu_1) = l(\nu_0)$. This implies that $\alpha_{\nu_0}|_{[0,l(\nu_0)]}$ is minimal, which is a contradiction, since we assumed that $\alpha_{\nu_0}|_{[0,l(\nu_0)]}$ is not minimal. Therefore, for any $\nu \in (m(t_0), m(u)]$, the geodesic segments $\alpha_\nu|_{[0,l(\nu)]}$ and $\beta_\nu|_{[0,l(\nu)]}$ are minimal geodesic segments joining q to the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu)) = \alpha_\nu(l(\nu))$. In particular, the point $\alpha_\nu(l(\nu)) = \beta_\nu(l(\nu))$ is a cut point of q. □

Proposition 4.9. *The cut locus of any point q with $|\tilde{t}(q)| < t_0$ equals the set*

$$\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}.$$

Here the coordinates $(\tilde{t}, \tilde{\theta})$ are chosen so as to satisfy $\tilde{\theta}(q) = 0$.

Proof. Without of loss of generality, we may assume that $-t_0 < \tilde{t}(q) < 0$. By Lemma 4.8, the geodesic segments $\alpha_\nu|_{[0,l(\nu)]}$ and $\beta_\nu|_{[0,l(\nu)]}$ are minimal for any $\nu \in (m(t_0), m(u)]$. Hence their limit geodesics $\alpha^- := \alpha_{m(t_0)}$ and $\beta^+ := \beta_{m(t_0)}$ are rays, that is, any their subarcs are minimal. Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{\theta} = 0$, it is trivial from Lemma 4.8 that the set $\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}$ is a subset of the cut locus of q. Suppose that there exists a cut point $y \notin \{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}$. Without loss of generality, we may assume that $\tilde{\theta}(y) > 0 = \tilde{\theta}(q)$. Since the cut locus of q has a tree structure, there exists an end point x of the cut locus in the set $\{(\tilde{t}, \tilde{\theta}) | \ \tilde{\theta} > 0\} \backslash D(\beta^+, \alpha^-)$, where $D(\beta^+, \alpha^-)$ denotes the closure of the unbounded domain cut off by β^+ and α^-. Hence, x is conjugate to q along any minimal geodesic segments γ joining q to x. Since such a minimal geodesic γ runs in the set $\{(\tilde{t}, \tilde{\theta}) \mid \tilde{\theta} > 0\} \setminus D(\beta^+, \alpha^-)$, by applying the Clairaut relation at the point q, we get that the Clairaut constant of γ is positive and less than $m(t_0)$. Notice that the geodesics β^+ and α^- have the same Clairaut constant $m(t_0)$. It follows from Lemma 4.5 that the geodesic γ cannot be tangent to any parallel arc. From Corollary 7.2.1 in [SST], γ has no conjugate point of q, which is a contradiction. □

Lemma 4.10. *Let q be a point in $\widetilde{M}$ with $|\tilde{t}(q)| = t_0$. Then, the cut locus of q is empty.*

Proof. We may assume that $\tilde{t}(q) = -t_0$, since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$. Supposing that there exists a cut point x of q, we will get a contradiction. Since $\widetilde{M}$ is simply connected, the cut locus has an end point. Hence, we may assume that the cut point x is an end point of C_q. Let γ be a minimal geodesic segment joining q to x. Then, x is a conjugate point of q along γ, since x is an end point of the cut locus. From Lemma 4.7, $\tilde{t}(x) \neq -t_0$. By applying the Clairaut relation, we obtain that the Clairaut constant of γ is smaller than $m(t_0)$, and hence γ is not tangent to any parallel arc by Lemma 4.6. Therefore, by Corollary 7.2.1 in [SST], there does not exist a conjugate point of q along γ, which is a contradiction. $\square$

Lemma 4.11. *Let q be a point in $\widetilde{M}$ with $|\tilde{t}(q)| > t_0$. If the Gaussian curvature of $\tilde{M}$ is nonpositive on $\tilde{t}^{-1}(-\infty, -t_0) \cup \tilde{t}^{-1}(t_0, \infty)$, then the cut locus of the point q is empty.*

Proof. Suppose that the cut locus of a point q with $|\tilde{t}(q)| > t_0$ is nonempty. Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$, we may assume that $\tilde{t}(q) < -t_0$. Supposing the existence of a cut point of q, we will get a contradiction. We may assume that there exists an end point x of C_q, since $\widetilde{M}$ is simply connected. Let $\gamma : [0, d(q, x)] \to \widetilde{M}$ be a unit speed minimal geodesic joining q to x. If $\tilde{t}(\gamma(s)) \leq -t_0$ for any $s \in (0, d(q, x)]$, then γ has no conjugate point of q, since the Gaussian curvature is nonpositive on $\tilde{t}^{-1}(-\infty, -t_0) \cup \tilde{t}^{-1}(t_0, \infty)$. This contradicts the fact that x is an end point of C_q. Thus we may assume that $\tilde{t}(\gamma(s)) > -t_0$ for some $s \in (0, d(q, x)]$. This implies that γ passes through a point of $\tilde{t}^{-1}(-t_0, t_0)$. It follows from the Clairaut relation that the Caliraut constant of γ is smaller than $m(t_0)$. Hence, from Corollary 7.2.1 in [SST] and Lemma 4.6, there does not exist a conjugate point of q along γ, which is a contradiction. $\square$

Proof of Main Theorem

Since the functions m and φ are decreasing on $[0, t_0)$ and $(m(t_0), m(0))$ respectively, the composite function $\varphi \circ m$ is increasing on $(0, t_0)$. It is clear to see that $\lim_{t \nearrow t_0} \varphi \circ m(t) = \infty$, since the minimal geodesic segment $\gamma_\nu|_{[0,l(\nu)]}$ converges to the ray $\gamma_{m(t_0)}$ as $\nu \searrow m(t_0)$. Let $t = t_\pi \in [0, t_0)$ be a solution of $\varphi \circ m(t) = \pi$. Define $t_\pi = 0$ if there is no solution. Hence, $\varphi \circ m(t) \geq \pi$ on $[t_\pi, t_0)$ and $\varphi \circ m(t) \leq \pi$ on $(0, t_\pi)$. Now the Main Theorem is clear from Proposition 4.9 and Lemma 4.10.

4.4 A family of cylinders of revolution

An example of a cylinder of revolution satisfying the two properties 1 and 2 in the introduction was given by Tamura [Ta]. The Riemannian metric ds^2 is defined by $ds^2 = dt^2 + e^{-t^2} d\theta^2$. It is easy to see that $m' = -2t \cdot m < 0$ on $(0, \infty)$, and the Gaussian curvature $G(q)$ at a point q is $-4t^2(q) + 2$. This implies that the Gaussian curvature is decreasing on each upper half meridian of the surface. By Lemma 4.5 in [C1], the cut locus of a point on $\tilde{t} = 0$ on the universal covering space of the surface is a nonempty subset of $\tilde{t} = 0$. Hence, this surface satisfies the assumptions of the Main Theorem. The following family of cylinders of revolution shows that the converse is not true, i.e., under the assumptions of the Main Theorem, the decline of the Gaussian curvature does not always hold.

In this section we give a family of cylinders of revolution $\{M_\lambda\}_\lambda := \{(R^1 \times S^1, dt^2 + m_\lambda(t)^2 d\theta^2)\}_\lambda$ satisfying the assumtions in the Main Theorem, where $\lambda > 1$ denotes a parameter and

$$m_\lambda(t) := \frac{\cosh t}{\sqrt{1 + \lambda \sinh^2 t}}. \tag{4.2}$$

Lemma 4.12. *The Gaussian curvature $G(q)$ at a point $q \in M_\lambda$ is given by*

$$G(q) = (\lambda - 1)\left(\frac{3}{h^2(t(q))} - \frac{2}{h(t(q))}\right), \tag{4.3}$$

where $h(t) = 1 + \lambda \sinh^2 t$. In particular, the Gaussian curvature G is not monotonic along the upper half meridian $\theta^{-1}(0) \cap t^{-1}(0, \infty)$.

Proof. From (4.2), we get

$$m'_\lambda(t) = (1 - \lambda) m_\lambda(t) \tanh t / h(t), \tag{4.4}$$

and

$$m''_\lambda(t) = ((1 - \lambda)\tanh t / h(t))^2\, m_\lambda(t) + (1 - \lambda) m_\lambda(t)(h(t)/\cosh^2 t - h'(t)\tanh t)/h(t)^2.$$

Thus, we obtain

$$m''_\lambda = (1 - \lambda) m_\lambda(t)((1 - \lambda)\tanh^2 t + h(t)/\cosh^2 t - h'(t)\tanh t)/h^2.$$

Since $(1 - \lambda)\tanh^2 t + h(t)/\cosh^2 t = 1$ holds, we have

$$-m''_\lambda(t)/m_\lambda(t) = (\lambda - 1)\left(3/h^2(t) - 2/h(t)\right).$$

Since $G(q) = -m''_\lambda(t(q))/m_\lambda(t(q))$, we obtain (4.3). By (4.4), it is trivial to see that the Gaussian curvature is not monotonic along the upper half meridian. $\square$

Lemma 4.13. *Let $a, b \in (0, 1)$ be numbers with $a < b$. Then,*

$$\int_b^1 \frac{dx}{x(x - a)\sqrt{(x - b)(1 - x)}} = \frac{\pi}{a(1 - a)}\left(\frac{a - 1}{\sqrt{b}} + \frac{1}{c}\right) \tag{4.5}$$

holds, where $c = \sqrt{(b - a)/(1 - a)}$.

Proof. From a direct computation, we obtain

$$\frac{d}{du}\left(\frac{a-1}{\sqrt{b}}\arctan\frac{u}{\sqrt{b}}+\frac{1}{c}\arctan\frac{u}{c}\right)=\frac{a-1}{u^2+b}+\frac{1}{u^2+c^2} \tag{4.6}$$

and

$$\frac{du}{dx}=\frac{(1-b)u}{2(x-b)(1-x)}, \tag{4.7}$$

where $u=\sqrt{(x-b)/(1-x)}$. Since $c^2=(b-a)/(1-a)$, we get

$$\frac{a-1}{u^2+b}+\frac{1}{u^2+c^2}=\frac{a(1-a)(1-x)}{(1-b)x(x-a)}. \tag{4.8}$$

By (4.6), (4.7) and (4.8), we have

$$\frac{d}{dx}\left(\frac{a-1}{\sqrt{b}}\arctan\frac{u}{\sqrt{b}}+\frac{1}{c}\arctan\frac{u}{c}\right)=\frac{a(1-a)}{2}\frac{1}{x(x-a)\sqrt{(x-b)(1-x)}}.$$

This implies that

$$\int\frac{dx}{x(x-a)\sqrt{(x-b)(1-x)}}=\frac{2}{a(1-a)}\left(\frac{a-1}{\sqrt{b}}\arctan\frac{u}{\sqrt{b}}+\frac{1}{c}\arctan\frac{u}{c}\right)$$

holds. Hence, we obtain (4.5). □

By (4.2) and (4.4), we get $\inf m_\lambda=1/\sqrt{\lambda}$ and $m'_\lambda(t)<0$ for any $t>0$. Hence the half period function $\varphi(\nu)$ for M_λ is defined on $(1/\sqrt{\lambda},1)$.

Lemma 4.14. *The half period function $\varphi(\nu)$ is given by*

$$\varphi(\nu)=\pi\left(-\sqrt{\lambda-1}+\frac{\lambda\nu}{\sqrt{\lambda\nu^2-1}}\right)$$

on $(\frac{1}{\sqrt{\lambda}},1)$. In particular φ is decreasing on $(\frac{1}{\sqrt{\lambda}},1)$ and the surface M_λ satisfies the assumptions of the Main Theorem.

Proof. By putting $x:=m_\lambda^2(t)$, we get, by (4.4),

$$dt=\frac{h(t)}{2(1-\lambda)x\tanh t}dx. \tag{4.9}$$

Since $x=(1+\sinh t^2)/h(t)$,

$$\sinh^2 t=\frac{1-x}{\lambda x-1},\quad \cosh^2 t=\frac{(\lambda-1)x}{\lambda x-1},\quad \text{and }\ h(t)=\frac{(\lambda-1)}{\lambda x-1}. \tag{4.10}$$

By combining (4.9) and (4.10), we obtain,

$$dt = \frac{-\sqrt{\lambda - 1}}{2(\lambda x - 1)\sqrt{x(1-x)}} dx. \tag{4.11}$$

Therefore, by (4.1),

$$\varphi(\nu) = \nu\sqrt{\lambda - 1} \int_{\nu^2}^{1} \frac{dx}{x(\lambda x - 1)\sqrt{(x - \nu^2)(1 - x)}}$$

for $\nu \in (1/\sqrt{\lambda}, 1)$. It follows from Lemma 4.13 that $\varphi(\nu) = \pi \left(-\sqrt{\lambda - 1} + \lambda\nu/\sqrt{\lambda\nu^2 - 1}\right)$. It is easy to check that $\varphi'(\nu) = -\pi \left(1/(2\sqrt{\lambda - 1}) + \lambda/\sqrt{\lambda\nu^2 - 1}^3\right) < 0$ on $(1/\sqrt{\lambda}, 1)$. Therefore, by Proposition 4.4, the surface M_λ satisfies the assumptions of the Main Theorem. $\square$

Chapter 5

Final remarks

We summerize the Chapter 3 and the Chapter 4. Let (M, ds^2) denote a complete Riemannian manifold $R^1 \times S^1$ with a warped product metric $ds^2 = dt^2 + m(t)^2 d\theta^2$ of the real line (R^1, dt^2) and the unit circle $(S^1, d\theta^2)$. Suppose that m is an even function, and the Gaussian curvature is positive on the equator $t = 0$. Then the half period function $\varphi(\nu)$ is defined on $(m(t_0), m(0))$, where $t_0 := \sup\{t > 0 | m'|_{(0,t)} < 0\}$, and $m(t_0) := \inf m$ if $t_0 = \infty$.

We have proved the following structure theorem of the cut locus for a certain class of cylinders.

Theorem. *If the half period function $\varphi(\nu)$ is decreasing on $(m(t_0), m(0))$, then there exists $t_\pi \in [0, t_0)$ such that for any point $q \in \theta^{-1}(0)$ with $|t(q)| < t_\pi$, the cut locus of q equals*

$$\theta^{-1}(\pi) \cup \big(t^{-1}(-t(q)) \cap \theta^{-1}[\varphi(m(t(q))), 2\pi - \varphi(m(t(q)))]\big)$$

and for any point q with $t_\pi \leqslant |t(q)| \leqslant t_0$ equals $\theta^{-1}(\pi)$, the meridian opposite to q.

In the Chapter 3, a sufficient condition for $\varphi(\nu)$ to be decreasing on $(m(t_0), m(0))$ is given: If the Gaussian curvature is decreasing on the upper half meridian $\theta^{-1}(0) \cap t^{-1}[0, \infty)$, and positive on the equator, then φ is decreasing on $(m(t_0), m(0))$, the Gaussian curvature is nonpositive on $t^{-1}(t_0, \infty)$, and $m(t_0) = \inf m$.

In the Chapter 4, a necessary and sufficient condition for $\varphi(\nu)$ to be decreasing on $(m(t_0), m(0))$ is also given by using the universal covering space of the cylinder.

The family $\{M_\lambda\}_{\lambda>1}$ of cylinders of revolution introduced in the Chapter 4 can be realized in Euclidean 3-space if $\lambda > 1$ satisfies $\sqrt{\lambda}(\lambda - 1) \leqslant 1$. In particular, if $\lambda \leqslant 1.7$, then M_λ is isometrically embedded in Euclidean 3-space. Incidentally, the half period function for M_λ is decreasing for each $\lambda > 1$, but the Gaussian curvature of M_λ is not monotone on the half meridian $\theta^{-1}(0) \cap t^{-1}[0, \infty)$.

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