

論文の内容の要旨

論文題目 「Maximal Diameter Sphere Theorem」
(最大直径球面定理に関する研究)

学位申請者 Nathaphon Boonnam

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It is known that the diameter of a complete surface is at most π if the Gaussian curvature is not less than 1. In the middle of 19th century, by applying the Sturm comparison theorem on two differential equations to the complete surface, this was proved by Bonnet. Intuitively, if a complete surface is curved than the unit sphere, then the diameter of the surface does not exceed π . In 1940's, Myers generalized this result to any n -dimensional complete Riemannian manifolds, that is, if the sectional curvature (more generally Ricci curvature) of the manifold is not less than 1 at each point, then the diameter of the manifold is at most π . In 1950's, Toponogov, a Russian geometrician, observed the case where the diameter equals π . He proved that if the diameter of a complete n -dimensional Riemannian manifold M equals π and the sectional curvature of the manifold is at least 1, then the manifold is isometric to the n -dimensional unit sphere.

Theorem. (Toponogov) *If the sectional curvature of the manifold M is at least 1, then the diameter of M is at most π and furthermore, if the diameter equals π , then M is isometric to the n -dimensional unit sphere.*

Incidentally, the theorem above is called a maximal diameter sphere theorem. Moreover, in 1970's, Cheng proved the theorem above under the assumption that the Ricci curvature is at least 1. Toponogov proved the maximal diameter sphere theorem by applying a theorem, which is called the Toponogov comparison theorem. This comparison theorem is now a necessary and useful tool in studying the global structure of Riemannian manifolds. Here we should introduce the Toponogov comparison theorem very shortly. When we draw a geodesic triangle on a complete

Riemannian manifold whose sectional curvature is at least 1 at each point, we can find a corresponding geodesic triangle on the unit sphere to the above one which has each side equals length to the corresponding one and the inner angle of each vertex is not greater than the inner angle of the corresponding one.

In the Chapter 1 of the present thesis, the history of the maximal diameter sphere theorem is introduced. Furthermore, some fundamental knowledge in Riemannian geometry for understanding the maximal diameter sphere theorem is stated and the proofs are given for some fundamental theorems. For example, the sectional curvature, the index form and its basic property are introduced and their proofs are given here.

In the Chapter 2, the proof of Cheng's maximal diameter sphere theorem is given. At first, the Bishop-Gromov theorem (the comparison theorem on the volume ratio) is introduced in order to prove Cheng's theorem. By making use of the Bishop-Gromov theorem, we can prove that the sectional curvature is equal to 1 everywhere. Thus, the manifold is isometric to the unit sphere. In addition, the proofs of the Sturm comparison

theorem and Rauch comparison theorem are given.

In the Chapter 3, the following maximal diameter sphere theorem proved by N.Boonnam, the author of the present thesis, is introduced and its proof is given. Instead of the unit sphere, the reference surface is a 2-sphere of revolution which has two common properties that an ellipsoid has.

Theorem. (Nathaphon Boonnam) *If an n -dimensional Riemannian manifold M is more curved than the references surface above, then the diameter of M is at most 1. Here the diameter of the reference surface is assumed to be π by scaling its Riemannian metric. Moreover, if the diameter equals π , then M is isometric to the n -dimensional reference space. In particular, if the reference surface is an ellipsoid, then M is isometric to the n -dimensional ellipsoid.*

A main tool for proving this theorem is a generalized Toponogov comparison theorem proved by Sinclair and Tanaka. The perimeter of any geodesic triangle is at most 2π and if the perimeter is 2π , then the geodesic triangle is a great circle. By supposing that the points attaining the diameter of the manifold M are not the base point, we get a contradiction. Thus, the pair of the base point and its furthest point attains the diameter of M . The generalized Toponogov comparison theorem plays a very important role. If the base point and its farthest points attain the diameter, then it is not difficult to construct an isometry between M and the unit sphere.

In Chapter 4, the detail proof of a corollary to this theorem is given. The reference surface is an ellipsoid. The base point of the n -dimensional ellipsoid is an intersection with the rotation axis. First, it is proved that the ellipsoid is of revolution. Next, it is proved that the radial planes at two points have the same sectional curvature if the two points are equidistant from the base point. Therefore, the sectional curvature of a radial plane is a function depending only the distance from the base point. It is proved that each 2-dimensional ellipsoid is totally geodesic in the n -dimensional ellipsoid, and that the n -dimensional reference space of a 2-dimensional ellipsoid is equal to the n -dimensional ellipsoid.

In the last Chapter, an open problem and some remarks on this problem are introduced. For example, in the case where the reference surface is the 2-dimensional unit sphere, that is, the radial curvatures are at most 1, it is not known if the manifold is isometric to the n -dimensional unit sphere.