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Conditional Multivariate One-Sided Test

by

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Abstract

In this study we construct a conditional test for multivariate one-sided test with known covariance matrix by referring to Kudo (1963) and Fraser, Guttman and Srivastava (1991). Then, we consider the power of the test for this conditional multivariate one-sided test. Finally, we compare the conditional multivariate one-sided test with Kudo's multivariate one-sided test in terms of the power of the test through numerical examples.
Key words: χ^2-distribution, orthogonal projection, power of the test

1. Introduction

Multivariate one-sided test which tests whether a normal mean vector with nonnegative components is equal to zero or not is used for testing the superiority and the inferiority between two treatments evaluated by multivariate normal response. Furthermore, it can be applied to testing differences in a sequence of normal means with ordered restriction. Kudo (1963) proposed a multivariate one-sided test with known covariance matrix based on likelihood ratio test. Specifically, Kudo (1963) derived the likelihood ratio test statistic and determined its distribution explicitly. On the other hand, Perlman (1969) considered a multivariate one-sided test with unknown covariance matrix. Perlman (1969) derived the approximate distribution of the likelihood ratio test statistic, since it is difficult to determine the exact distribution. Wang and McDermott (1998) derived the conditional distribution of the likelihood ratio test statistic given the sufficient statistic for unknown covariance matrix. However, these tests are accompanied with theoretical and computational complications. Specifically, we have to determine the orthogonal projection of the observed point on the positive quadrant for determining the likelihood ratio test statistic, and the formulation of orthogonal projection and the distribution of the statistic vary with the location of observed point. Yamamoto, Kudo and Ujiie (1997) constructed a handy method for computing the test statistic and its significance probability based on applying sweep out operations on a certain matrix in a systematic manner and applying the Fortran subroutine of Sun (1988). On the other hand, many researchers proposed simple procedures for multivariate one-sided test by devising hypotheses and statistics of Kudo (1963) or Perlman (1969). Fraser, Guttman and Srivastava (1991) proposed a multivariate one-sided test with unknown covariance matrix under the condition that the orthogonal projection of the observed point is determined. Therefore, the conditional likelihood ratio test statistic is distributed according to an F-distribution uniquely.

On the other hand, the idea of Fraser, Guttman and Srivastava (1991) can be applied to Kudo's multivariate one-sided test. Although the assumption that the covariance matrix is known seems impractical, it is occasionally known that the covariance matrix is constant through the previous experiments. Even if the covariance matrix is unknown, we can use a sample covariance matrix constructed by a large number of observations instead of the unknown covariance matrix. The aim of this study is to construct a conditional multivariate one-sided test with known covariance matrix by referring to Kudo (1963) and Fraser, Guttman and Srivastava (1991). Specifically, we consider a multivariate one-sided test under the condition that the orthogonal projection of the observed point is determined. Therefore, the conditional likelihood ratio test statistic is distributed according to a χ^2-distribution uniquely. Then, we consider the power of the test for this conditional multivariate one-sided test. Finally, we compare our conditional multivariate one-sided test with Kudo's multivariate one-sided test in terms of the power of the test through numerical examples in a simple situation.

2. Kudo's multivariate one-sided test

First, we discuss the multivariate one-sided test based on the likelihood ratio test proposed by Kudo (1963). We summarize it without any proofs.

2.1. Likelihood ratio test

Assume p-dimensional random variable vector$\boldsymbol{X}$ is distributed according to multivariate normal $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with unknown mean vector $\boldsymbol{\mu}$ and known covariance matrix $\boldsymbol{\Sigma}$. Letting $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$, assume

$$\mu_1 \geq 0,\ \mu_2 \geq 0,\ \ldots,\ \mu_p \geq 0. \tag{2.1}$$

If $\mu_i > 0$ for at least one i in (2.1), we denote

$$\boldsymbol{\mu} \geq^* \mathbf{0}.$$

We set up a null hypothesis versus its alternative hypothesis as

$$H_0 : \boldsymbol{\mu} = \mathbf{0} \ \text{ v.s. } \ H_1 : \boldsymbol{\mu} \geq^* \mathbf{0}.$$

For a sample $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, let

$$\bar{\boldsymbol{X}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i, \quad \boldsymbol{Z} = \sqrt{n}\bar{\boldsymbol{X}}.$$

We denote the orthogonal projection of $\boldsymbol{Z}$ onto

$$\boldsymbol{\Theta} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_p)' \in \boldsymbol{R}^p | x_1 \geq 0, x_2 \geq 0, \ldots, x_p \geq 0\}$$

with regard to Mahalanobis distance $\| \cdot \|_{\boldsymbol{\Sigma}}$ based on $\boldsymbol{\Sigma}$ by $\pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z}, \boldsymbol{\Theta})$. By the likelihood ratio test criteria derived by Kudo (1963) we reject H_0 when

$$\bar{\chi}^2 = \| \pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z}, \boldsymbol{\Theta}) \|_{\boldsymbol{\Sigma}}^2 > c \tag{2.2}$$

for a specified critical value c.

2.2. Determination of the orthogonal projection and the distribution of the likelihood ratio test statistic

First we discuss how to determine the orthogonal projection $\pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z},\boldsymbol{\Theta})$. We consider the following simple decomposition

$$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{Z}_1 \\ \boldsymbol{Z}_2 \end{pmatrix}. \tag{2.3}$$

Herein $\boldsymbol{Z}_2$ is an m-dimensional vector. Let

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

be the decomposition corresponding to (2.3). Therefore, $\boldsymbol{\Sigma}_{22}$ is an $m \times m$ matrix. If

$$\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{Z}_1 \leq \boldsymbol{0}, \quad \boldsymbol{Z}_{2\cdot 1} = \boldsymbol{Z}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{Z}_1 > \boldsymbol{0}, \tag{2.4}$$

the orthogonal projection is determined as

$$\pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z},\boldsymbol{\Theta}) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{Z}_{2\cdot 1} \end{pmatrix}. \tag{2.5}$$

Herein $\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{Z}_1 \leq \boldsymbol{0}$ means that all components of $\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{Z}_1$ are nonpositive, and $\boldsymbol{Z}_{2\cdot 1} = \boldsymbol{Z}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{Z}_1 > \boldsymbol{0}$ means that all components of $\boldsymbol{Z}_{2\cdot 1}$ are positive. Then

$$\bar{\chi}^2 = \| \pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z},\boldsymbol{\Theta}) \|_{\boldsymbol{\Sigma}}^2 = \boldsymbol{Z}_{2\cdot 1}'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{Z}_{2\cdot 1}, \tag{2.6}$$

where

$$\boldsymbol{\Sigma}_{22\cdot 1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}.$$

Under $H_0 : \boldsymbol{\mu} = \boldsymbol{0}$,

$$\bar{\chi}^2 \sim \chi_m^2$$

independently of the event (2.4).

In general case, if the decomposition

$$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{Z}_1 \\ \boldsymbol{Z}_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

obtained by permuting components appropriately satisfies (2.4), $\bar{\chi}^2$ is determined by (2.5) and (2.6).

2.3. Determination of the critical value for a specified significance level

By the discussion in the previous subsection $\bar{\chi}^2 \sim \chi^2_m$, if and only if the number of positive components of $\pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z}, \boldsymbol{\Theta})$ is m. Therefore, denoting the probability that the number of positive components of $\pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z}, \boldsymbol{\Theta})$ is m by $P(m)$, we obtain

$$P(\bar{\chi}^2 > c) = \sum_{m=1}^{p} P(m) P(\chi^2_m > c). \tag{2.7}$$

Then, we determine the critical value c so that (2.7) may be equal to a specified significance level α. However, it is difficult to calculate $P(m)$ for large p.

3. Conditional multivariate one-sided test

In the previous section we discussed the multivariate one-sided test proposed by Kudo (1963). This test is accompanied with some theoretical and computational complications in determining the critical value satisfying a specified significance level when p is large. Therefore, various simple methods for multivariate one-sided test have been proposed. Fraser, Guttman and Srivastava (1991) proposed a multivariate one-sided test with unknown covariance matrix under the condition that the orthogonal projection of the observed point is determined. Therefore, the likelihood ratio test statistic is distributed according to an F-distribution uniquely. We consider the similar conditional test for Kudo's multivariate one-sided test.

Under the condition that the orthogonal projection of $\boldsymbol{Z}$ is determined as (2.5) through (2.4) for decomposition (2.3), $\bar{\chi}^2 \sim \chi^2_m$ where m is the dimension of $\boldsymbol{Z}_2$. Then, the critical value of this conditional multivariate one-sided test for a specified significance level α can be determined easily as the percent point of χ^2_m-distribution. We denote this critical value by c_m. By this conditional test we reject H_0 when $\bar{\chi}^2 > c_m$.

4. Power of the test

Assuming p-dimensional vector $\boldsymbol{\mu}^*$ satisfies $\boldsymbol{\mu}^* \geq^* \boldsymbol{0}$, we consider the power of the test under alternative hypothesis $H_1 : \boldsymbol{\mu} = \boldsymbol{\mu}^*$.

4.1. The power of Kudo's multivariate one-sided test

We denote the decomposition

$$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{Z}_1 \\ \boldsymbol{Z}_2 \end{pmatrix} \tag{4.1}$$

by

$$\Lambda(\boldsymbol{Z}_1, \boldsymbol{Z}_2). \tag{4.2}$$

Abbreviating $\Lambda(\boldsymbol{Z}_1, \boldsymbol{Z}_2)$ to Λ, we denote the dimension of $\boldsymbol{Z}_2$ for decomposition (4.2) by $\dim\Lambda$. If

$$\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{Z}_1 \leq \boldsymbol{0}, \quad \boldsymbol{Z}_{2\cdot1} = \boldsymbol{Z}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{Z}_1 > \boldsymbol{0} \tag{4.3}$$

for (4.2),

$$\pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z}, \boldsymbol{\Theta}) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{Z}_{2\cdot1} \end{pmatrix}.$$

We denote the probability that (4.3) occurs for $\Lambda = \Lambda(\boldsymbol{Z}_1, \boldsymbol{Z}_2)$ under $H_1 : \boldsymbol{\mu} = \boldsymbol{\mu}^*$ by $P(\Lambda)$. On the other hand, let

$$\boldsymbol{\mu}^* = \begin{pmatrix} \boldsymbol{\mu}_1^* \\ \boldsymbol{\mu}_2^* \end{pmatrix}$$

be the decomposition of $\boldsymbol{\mu}^*$ corresponding to Λ. Then $\bar{\chi}^2 = \boldsymbol{Z}_{2\cdot1}'\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{Z}_{2\cdot1}$ is distributed according to noncentral χ^2-distribution with $\dim\Lambda$ degrees of freedom and noncentrality $u(\Lambda) = n\boldsymbol{\mu}_{2\cdot1}^{*'}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\mu}_{2\cdot1}^*$, where $\boldsymbol{\mu}_{2\cdot1}^* = \boldsymbol{\mu}_2^* - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1^*$. We denote this distribution by $\chi^2_{\dim\Lambda, u(\Lambda)}$. Then the power is given by

$$P(\bar{\chi}^2 > c) = \sum_{\Lambda} P(\Lambda) P(\chi^2_{\dim\Lambda, u(\Lambda)} > c).$$

4.2. The power of conditional multivariate one-sided test

Next, we consider the power of conditional multivariate one-sided test discussed in Section 3. Since the critical value is determined by $\dim\Lambda$ for decomposition (4.2) and denoted by $c_{\dim\Lambda}$, the power under $H_1 : \boldsymbol{\mu} = \boldsymbol{\mu}^*$ is

$$\sum_{\Lambda} P(\Lambda) P(\chi^2_{\dim\Lambda, u(\Lambda)} > c_{\dim\Lambda}). \tag{4.4}$$

5. Power comparison through numerical examples

In this section we give some numerical examples in a simple situation intended to compare the conditional multivariate one-sided test with Kudo's multivariate one-sided test in terms of the power of the test.

Let $p = 2$ and

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Table 1 gives critical values of Kudo (1963) for significance level $\alpha = 0.05$ and $\rho = -0.9, -0.6, -0.3, 0, 0.3, 0.6, 0.9$.

Table 1: Critical value c of Kudo's multivariate one-sided test ($\alpha = 0.05$)

ρ	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
c	0.620	2.380	3.650	4.240	4.050	2.980	0.940

On the other hand, the critical values of conditional multivariate one-sided test are given by $c_1 = 3.84, c_2 = 5.99$ corresponding to $\dim\Lambda = 1, 2$.

We calculate the power of the test by using these critical values. Let $n = 30$. Table 2~7 give the power under the alternative hypotheses $H_1\colon \boldsymbol{\mu} = (\delta, \delta)'$, $H_1\colon \boldsymbol{\mu} = (\delta, 0)'$ for $\delta = 0.2, 0.4, 0.6$. Here O-S expresses Kudo's multivariate one-sided test and C-O-S expresses the conditional multivariate one-sided test.

The power under $H_1\colon \boldsymbol{\mu} = (\delta, \delta)'$ decreases as ρ is higher. The power of O-S is higher than C-O-S for each δ and ρ. However, when δ is large and ρ is low, the difference between O-S and C-O-S is small. The power under $H_1\colon \boldsymbol{\mu} = (\delta, 0)'$ decreases as ρ varies from -0.9 to 0, then it increases as ρ varies from 0 to 0.9. The power of O-S is higher than C-O-S for each δ and ρ. However, when δ is large and ρ is low or high, the difference between O-S and C-O-S is small.

Table 2: Power under $H_1\colon \boldsymbol{\mu} = (0.2, 0.2)'$

ρ	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
O-S	0.999	0.764	0.536	0.405	0.323	0.267	0.219
C-O-S	0.997	0.626	0.395	0.287	0.227	0.189	0.163

Table 3: Power under H_1: $\boldsymbol{\mu} = (0.4, 0.4)'$

ρ	−0.9	−0.6	−0.3	0	0.3	0.6	0.9
O-S	0.999	0.999	0.970	0.892	0.794	0.670	0.601
C-O-S	0.999	0.996	0.926	0.804	0.685	0.587	0.509

Table 4: Power under H_1: $\boldsymbol{\mu} = (0.6, 0.6)'$

ρ	−0.9	−0.6	−0.3	0	0.3	0.6	0.9
O-S	0.999	0.999	0.999	0.997	0.984	0.955	0.909
C-O-S	0.999	0.999	0.999	0.991	0.963	0.919	0.865

Table 5: Power under H_1: $\boldsymbol{\mu} = (0.2, 0.0)'$

ρ	−0.9	−0.6	−0.3	0	0.3	0.6	0.9
O-S	0.789	0.343	0.250	0.218	0.218	0.265	0.654
C-O-S	0.680	0.254	0.189	0.174	0.186	0.246	0.639

Table 6: Power under H_1: $\boldsymbol{\mu} = (0.4, 0.0)'$

ρ	−0.9	−0.6	−0.3	0	0.3	0.6	0.9
O-S	0.999	0.820	0.660	0.600	0.619	0.753	0.998
C-O-S	0.998	0.737	0.580	0.539	0.579	0.735	0.998

Table 7: Power under H_1: $\boldsymbol{\mu} = (0.6, 0.0)'$

ρ	−0.9	−0.6	−0.3	0	0.3	0.6	0.9
O-S	0.999	0.988	0.937	0.906	0.923	0.979	0.999
C-O-S	0.999	0.974	0.905	0.877	0.905	0.975	0.999

6. Conclusions

In this study we have discussed a simple multivariate one-sided test with known covariance matrix. Specifically, we constructed a conditional multivariate one-sided test by referring to Fraser, Guttman and Srivastava (1991). Although Kudo's multivariate one-sided test is accompanied with some theoretical and computational complications, our conditional multivariate one-sided test is a simple procedure with unique distribution. We confirmed that the conditional multivariate one-sided test is not powerful compared to Kudo's multivariate one-sided test through simple numerical examples. However, the difference between two tests is small in some situations. We should give numerical examples for power comparison in general cases. Furthermore, we should consider various applications of our simple multivariate one-sided test.

There remain some problems to be solved regarding our conditional multivariate one-sided test. First, we did not revise the method for determining the orthogonal projection $\pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z}, \boldsymbol{\Theta})$. We gave only numerical examples for $p = 2$. It is difficult to determine $\pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z}, \boldsymbol{\Theta})$ for large p. Yamamoto, Kudo and Ujiie (1997) constructed a method for computing the test statistic and its significance probability. We want to improve the method for determining $\pi_{\boldsymbol{\Sigma}}(\boldsymbol{Z}, \boldsymbol{\Theta})$ by referring to Yamamoto, Kudo and Ujiie (1997). On the other hand, we should improve the method for calculating the power of the test. Specifically, it is difficult to calculate the probability $P(\Lambda)$ for the decomposition Λ of $\boldsymbol{Z}$ under the alternative hypothesis. Miwa *et al.* (2003) considered the calculation of non-centred orthant probabilities which can be applied to the calculation of the power. We want to improve the method for calculating the power by referring to Miwa *et al.* (2003).

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