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CS-Aronszajn trees under the negation of the Continuum Hypothesis

by

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Introduction

Aronszajn tree T is called CS if every club subset S of T contains a stationary antichain, where a subset S of T is called club (resp. stationary) if the set of the heights of its all members forms a club (resp. stationary) subset of ω_1. This notion was introduced in [6] as a variation of anti-Suslin property (introduced in Baumgartner [1] under the name non-Suslin: see also Baumgartner [3]). Several trees that witness the difference between CS and certain similar properties were constructed there.

The existence of those trees was however shown in the constructible universe (in fact, a somewhat more general universe, but in which CH (the Continuum Hypothesis) holds). It is thus natural to ask its existence in the universe where CH fails. The present paper answers the question: the answer is affirmative.

§ 1. Basic notions

Our terminology is standard. The extra notions treated in this paper are as follows:

The height of the node x of a tree is denoted by $ht(x)$. A tree T is anti-suslin if every uncountable subset of T includes an uncountable anti-chain. A subset S of a tree is called *club* (resp. *stationary*) if the set $ht``(S)$ is a club (resp. stationary) in ω_1. A tree T is said to have a property CS (resp. SS) if every club (resp. stationary) subset of T includes a stationary anti-chain. A tree T is R-embeddable if there is an order homomorphism from T into $\mathbf{R}$. Similarly Q-embeddability is defined. By P_κ, we denote the Cohen forcing poset $\{p \mid p$ is a finite partial function from κ to $2\}$. T^* denotes the tree which has a field $\cup\{\mathbf{Q}^\alpha \mid \alpha\in\omega_1\}$ and is ordered by function-extension; i. e. $f<_T g$ iff $f\subsetneqq g$. When saying T is a subtree of T^*, we assume tacitly that T is an initial segment of T^*; i. e. if $x\in T$, all elements less than x in T^* belong to T.

The following are technical tools serving the proofs in last two sections. A sequence $\langle S_\alpha \mid \alpha<\omega_1\rangle$ is called an ω_1-hierarchy sequence, if (1) every S_α is of power ω_1 or less, and (2) $S_\alpha\subseteq S_\beta$ for $\alpha<\beta$, (3) $S_\beta=\cup\{S_\alpha \mid \alpha<\beta\}$ for limit β. For such a sequence, if $\diamondsuit$ holds, we can obtain a sequence $\langle \phi_\lambda \mid \lambda\in\omega_1\cap \mathrm{Lim}\rangle$ that has the following property:

(1) whenever the set X satisfies that $S_\alpha\cap X$ is countable for all $\alpha<\omega_1$, then there are stationary-many λ satisfying $\phi_\lambda=X\cap S_\lambda$.

This sequence is called a $\diamondsuit$-sequence for $\langle S_\alpha \mid \alpha<\omega_1\rangle$. Moreover, if $\diamondsuit^*$ holds, there is a sequence $\langle \Phi_\lambda \mid \lambda\in\omega_1\cap \mathrm{Lim}\rangle$ such that:

(1) Φ_λ is a countable family of sets,

(2) whenever the set X satisfies that $S_\alpha\cap X$ is countable for all $\alpha<\omega_1$, then for club-many ordinals λ, $X\cap S_\lambda$ belongs to Φ_λ.

This sequence is called a $\diamondsuit^*$-sequence for $\langle S_\alpha \mid \alpha<\omega_1\rangle$.

§ 2. Results about CS property

Let QE (resp. RE, CS, SS,UU) denote the class of Q-embeddable (resp. R-embeddable, CS, SS, anti-suslin) ω_1-trees. Then we have the following chain (see [1] and [6]):

$$\emptyset\subseteq \mathrm{QE}\subseteq \mathrm{RE}\cap \mathrm{CS}\subseteq \mathrm{CS}\subseteq \mathrm{RE}\cup \mathrm{CS}\subseteq \mathrm{UU}.$$

Also obvious is $\mathrm{QE}\subseteq \mathrm{SS}\subseteq \mathrm{CS}$. The question whether QE=SS is still open, but we do not touch this problem in this paper. If we assume $\diamondsuit^*$, we see that all the above inclusions are strict ([6]), i. e.:

$$\emptyset\neq \mathrm{QE}\subsetneqq \mathrm{RE}\cap \mathrm{CS}\subsetneqq \mathrm{CS}\subsetneqq \mathrm{RE}\cup \mathrm{CS}\subsetneqq \mathrm{UU}.$$

As is well-known, the asumption $\diamondsuit^*$ implies CH. So a natural question arises. The main purpose of this paper is to answer the question by proving the following:

The strict increasingness of the chain is compatible with the negation of CH.

There is no problem in QE$\neq\emptyset$, which is a theorem in ZF. The others need some arguments. Throughout this section, $V[G]$ means the generic universe obtained by the Cohen poset P_κ. We first note the following:

Theorem 2.1 ([5]) *if $\diamondsuit^*$ in V, there is an ω_1-tree in $V[G]$ which is R-embeddable, collectionwise hausdorff and not countably paracompact.*
It is known (Devlin and Shelah [4]) that if T is collectionwise hausdorff, then T has no stationary antichain, and a fortiori T is not CS. Thus:

Corollary 2.2 ($\diamondsuit^*$) CS$\subsetneqq$RE$\cup$CS *in* $V[G]$.
Next, we have the following, which will be proved in Section 3:

Theorem 2.3 ($\diamondsuit^*$)(RE$\cap$CS)\SS$\neq\emptyset$ *in* $V[G]$.

Corollary 2.4 ($\diamondsuit^*$) QE$\subsetneqq$RE$\cap$CS *in* $V[G]$.
When T is a tree, let T^Δ denote the tree whose field is $\{\langle x,0\rangle \mid x\in T|\,\mathrm{Lim}\}\cup\{\langle x,1\rangle \mid x\in T\}$ and whose order is defined by $\langle x,i\rangle<\langle y,j\rangle \leftrightarrow [x<y\vee[x=y\wedge i<j]]$.
Then the facts below hold: one is well known and the others are obvious.

Theorem 2.5 (1) (Baumgartner [2], see Shelah [7]) *T^Δ is R-embeddable, iff T is Q-embeddable,*
(2) *T is CS, iff T^Δ is CS,*
(3) *T is SS, iff T^Δ is SS.*
Hence, if T is in CS\SS, then T^Δ is in CS\(RE$\cup$SS). Thus, from the result mentioned above we obtain:

Theorem 2.6 ($\diamondsuit^*$) (1) CS\(RE$\cup$SS)$\neq\emptyset$ *in* $V[G]$,
(2) RE$\cap$CS$\subsetneqq$CS *in* $V[G]$.
As far as the second assertion concerns, the assumption $\diamondsuit^*$ can be weakened to $\diamondsuit$; we shall show this in Section 4. If T_1 is in RE\CS and T_2 is in CS\RE, then the simple union of them forms a tree which is in UU\(RE$\cup$CS), so:

Theorem 2.7 ($\diamondsuit^*$)RE$\cup$CS$\subsetneqq$UU *in* $V[G]$.
Thus our assertion is assured.

§ 3. CS-but not SS-Aronszajn tree in the Cohen extension

Assuming $\diamondsuit^*$, we construct an R-embeddable, CS, non-SS tree T which remains to have those properties in the Cohen generic extension $V[G]$. Obviously R-embeddability is preserved under the extension; the problems are in CS and SS.

For this purpose, we show the next lemma:

Lemma 3.1. ($\diamondsuit^*$) *Let E be a stationary and costationary subset of ω_1. There is an Aronszajn tree T and is an order-homomorphism $f : T \to \mathbf{R}$ such that :*

(1) $f``(T \mid (\omega_1 \backslash E)) \subseteq \mathbf{Q}$,

(2) *the statement "for all antichains A of $\check{T}$, $ht``(A) \cap \check{E}$ is non-stationary" is forced under P_{ω_1}.*

Before proving, we remark that the tree T in this lemma satisfies the final requirements. Let κ be any uncountable cardinal and $V[G]$ the generic extension by P_κ. First we note that:

(1) $\omega_1 \backslash E$ is stationary in V[G], by c.c.c. of P_κ,

(2) f maps $T \mid (\omega_1 \backslash E)$ into $\mathbf{Q}$ preserving the ordering.

So T is CS in $V[G]$. Next T has no stationary antichain included in $T \mid E$ in $V[G]$. For, if A is an antichain of T in $V[G]$, then for some ω_1-size subset H of κ the set A belongs to the model $V[G \cap (P_\kappa \mid H)]$. In this model, A does not meet with $T \mid E$ in stationary many levels by the property of T, and a fortiori so in $V[G]$. Hence T is not SS in $V[G]$.

Thus it remains only to prove the lemma.

Proof. Let P denote the poset P_{ω_1}. Recall T^* in Section 1. Fix a $\diamondsuit^*$-sequence $\langle \Phi_\alpha \mid \alpha \in \omega_1 \cap \mathrm{Lim} \rangle$ for a hierarchy sequence $\langle (T^* \mid \alpha) \times (P \mid \alpha) \mid \alpha < \omega_1 \rangle$.

Fix any stationary and costationary subset E of ω_1.

We give T_α and $f \mid T_\alpha$ by induction on α so that

(1) T_α is countable subset of $T_\alpha{}^*$,

(2) f is order preserving on $T|\alpha \cup T_\alpha$,

(3) for every pair of a node x in $T \mid \alpha$ and a positive rational b, it can be found an extension y of x in T_α such that $f(x) < f(y) < f(x) + b$.

(4) if α is in $\omega_1 \backslash E$, then $f(y)$ is a rational for all y in T_α.

(5) if α is a limit and in E, then for every $y \in T_\alpha$, for every member ϕ in Φ_α, for every $p \in P \mid \alpha$, there is a node x with $x <_T y$ such that

either (i) $\langle x, q\rangle \in \phi$ for some extension q of p.

or (ii) there is a rational a with $a > f(y)$ such that whenever $z >_T x$ with $z \in T \mid \alpha$ and $\langle z, q\rangle \in \phi$ for some extension q of p, then $f(z) \geq a$;

The node x that is described in the condition (5) is called a (p, ϕ)-search point for y.

The construction can be done naturally. We only present it only in the case that α is a limit ordinal in E. The core of the method is due to Devlin and Shelah[4].

For each pair of $x \in T \mid \alpha$ and $a \in \mathbf{Q}^+$, we pick out a certain node y from T_α^*, and T_α is defined to be the set of all those picked out y, where the f-value of y is defined by $f(y) = \sup\{f(z) \mid z <_T y\}$.

The node y is given as a unique extension of the branch determined by the sequence cofinal in $T \mid \alpha$ that is defined as follows:

The sequence, $\langle x_n \mid n < \omega\rangle$, starts from the node x, and will be built up by extending the nodes step by step in $T \mid \alpha$ in the following manner:

After fixing the ω-type enumeration of $P \mid \alpha \times \Phi_\alpha$, at each step n, we take the n-th pair $\langle p, \phi\rangle$ from $P \mid \alpha \times \Phi_\alpha$, and search for the extension z of the present node x_n that satisfies the following two conditions:

(1) $\langle z, q\rangle \in \phi$ for some extension q of p

(2) the value $f(z)$ is smaller than such a rational b_n as determined by: if $n=0$, $b_n = f(x) + a$; otherwise b_n is the mean value of $f(x_n)$ *and* b_{n-1}.

If there is no such z, then we put $z = x_n$. In any sufficient high level we can find an extension of z whose f-value is still less than b_n, which we take as x_{n+1}.

T_α is thus defined.

Now we show that the tree is not SS in the extension. Let $\dot{A}$ be a P-name such that $\Vdash$ "$\dot{A}$ is a maximal antichain of $\check{T}$". Put $A = \{\langle x, p\rangle \in T \times P \mid p \Vdash \check{x} \in \dot{A}\}$.

By the property of $\langle \Phi_\alpha \mid \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ there is a club set C_0 such that for every its element α, $A \cap ((T \mid \alpha) \times (P \mid \alpha))$ belongs to Φ_α. Let $\dot{C}$ be a P-name such that $\Vdash$ "$\dot{C}$ is the club set $\{\alpha \in \omega_1 \mid$ for all positive rational b and for all node $x \in \check{T} \mid \alpha$, if there is y in $\check{T}$ such that $(x <_T y \wedge y \in \dot{A} \wedge \check{f}(y) < b)$ then there is such y in $\check{T} \mid \alpha\}$"

Then there is a club set C_1 such that $\Vdash$ $\check{C}_1 \subseteq \dot{C}$. Let C_2 be a club set such that every its member α satisfies that whenever x is in $T \mid \alpha$, every $p \in P \mid \alpha$ has its extension q in $P \mid \alpha$ which decides the statement "$\check{x} \in \dot{A}$".

Let E_1 be the meet of E and the club $C_0 \cap C_1 \cap C_2$. We will show that $\Vdash$ "$\dot{A} \cap (\check{T} \mid \check{E}_1) = \emptyset$", which means "$\dot{A} \cap (T \mid E)$ is not stationary", so, the stationary-size set $T \mid E$ contains no stationary antichain even in the forcing extension.

Suppose to the contrary that there are some $\alpha\in E_1$ and some $y\in T_\alpha$ such that some condition r forces $\check{y}\in\dot{A}$. Put $p=r\restriction\alpha$. Note that the set $\phi=\{\langle v,\ q\rangle\in(T\restriction\alpha\times P\restriction\alpha)\mid q\Vdash\check{v}\in\dot{A}\}$ is in Φ_α since α is in C_0.

From the definition of T_α for $\alpha\in E\cap\mathrm{Lim}$, there is a $(p,\ \phi)$-search point x for the node y. There are two cases.

First case. There is an extension q of p such that $\langle x,\ q\rangle\in\phi$ for some $x<_T y$. Then q is an extension of $r\restriction\alpha$ and in $P\restriction\alpha$, so q and r have a common extension which forces "$\check{x}$, $\check{y}\in\dot{A}$", but $x<_T y$, which contradicts the antichain property of $\dot{A}$.

Second case. There is a rational $a>f(y)$ such that whenever a node $z>_T x$ and an extension $q\in P\restriction\alpha$ of p satisfies $\langle z,\ q\rangle\in\phi$, then $f(z)\geq a$. Since $\check{\alpha}\in\dot{C}$, the fact "$\check{y}\in\dot{A}\wedge\check{y}>_T x\wedge\check{f}(\check{y})<\check{a}$" implies the statement "there is $z\in\check{T}\restriction\alpha$ such that $z\in\dot{A}\wedge z>_T x\wedge\check{f}(z)<\check{a}$" in the forcing extension. Hence there are some node $z\in T\restriction\alpha$ and some extension $s\in P$ of r such that $z>_T x$, $f(z)<a$, and $s\Vdash\check{z}\in\dot{A}$. Since $\alpha\in C_2$, $s\restriction\alpha\Vdash\check{z}\in\dot{A}$, i. e. $\langle z,\ s\restriction\alpha\rangle\in\phi$. Thus $s\restriction\alpha\supseteq r\restriction\alpha=p$, $z>_T x$ and $f(z)<a$, which contradicts the case assumption. Lemma is thus proved.

§ 4. Non-R-embeddable CS-tree in the Cohen extension

As a corollary of the result of the previous section, we can see that the existence of a non-R-embeddable CS-tree is compatible with the negation of CH. The model is given by the Cohen extension of the ground model satisfying Jensen's principle $\Diamond^*$. For technical interest, we show under $\Diamond$ that such a tree exists in P_κ-generic extension $V[G]$. By an argument similar to that of the previous section, we can reduce the problem to the case $\kappa=\omega_1$. Hereafter P stands for P_{ω_1}. The following suffices for our purpose.:

Lemma 4.1. ($\Diamond$) *There is a tree T and a function $f:T\to\mathbf{Q}$ such that*

(1) $\Vdash_P$ *"T is not R-embeddable",*

(2) *there is a stationary $E\subseteq\omega_1$ satisfying $f``(T\restriction E)\subseteq\mathbf{Q}^+$,*

(3) *whenever a node $x\in T$ satisfies $f(x)>0$, there is $y<_T x$ such that f is strictly increasing on the interval $\{z\in T\mid y\leq_T z\leq_T x\}$.*

It is easy to see that the CS-ness follows from the conditions (2) and (3), and the both conditions are preserved in the forcing extension, so we obtain $\Vdash$ "$\check{T}$ is a CS-tree".

Proof. Let $\langle \phi_\lambda \mid \lambda \in \omega_1 \cap \mathrm{Lim}\rangle$ be a $\Diamond$-sequence for an ω_1-hierarchy sequence $\langle (T^* \mid \alpha) \times \mathbf{Q} \times (P \mid \alpha) \mid \alpha \in \omega_1\rangle$. We shall construct a subtree T of T^* and a function $f : T \to \mathbf{Q}^+ \cup \{0\}$ by induction on the levels of the tree so that at each α-th level the following conditions are guaranteed:

(1) T_α is countable,

(2) For each $b \in \mathbf{Q}^+$, every node x in $T \mid \alpha$ has an extension y in T_α satisfying that $f(y) \leq f(x) + b$ and f is strictly increasing from x to y,

(3) if $f(y) > 0$ for a node $y \in T_\alpha$, then there exists a node $x <_T y$ such that f is strictly increasing on the interval $[x, y]$,

(4) if α is not a limit or if $\phi_\alpha = \emptyset$, then all nodes y in T_α satsfy $f(y) > 0$,

(5) there is at most one node $y \in T_\alpha$ that satisfies $f(y) = 0$,

(6) if α is a limit and $\phi_\alpha \neq \emptyset$, there is a node w in T_α with the following property:

$\forall a \in \mathbf{Q}^+ \forall p \in P \mid \alpha \exists x <_T w$ such that

either (i) $\exists q \in P \mid \alpha (q$ extends $p \wedge \langle x, a, q\rangle \in \phi_\alpha)$

or (ii) $\forall q \in P \mid \alpha \forall z \in T \mid \alpha (q$ extends $p \wedge z >_T x \to \langle z, a, p\rangle \notin \phi_\alpha)$.

The construction is routine. We only mention how to obtain the node w described in Condition (6). First construct an ascending sequence $\langle x_n \mid n \in \omega\rangle$ cofinal in $T \mid \alpha$ so that x_0 $=$the root and each x_{2n+1} is chosen (if possible) so that for some condition $q \supset p$, $\langle x_{n+1}, a, q\rangle$ lies in ϕ_α, where $\langle a, q\rangle$ is the n-th element of the countable set $\mathbf{Q} \times P \mid \alpha$. Second let w be such a node in T_α^* that extends $\{x_n \mid n < \omega\}$. Then we let w be the unique enement of T_α satisfying $f(w) = 0$, and we call it the extra element of T_α. By a (p, a)-search point for w, we refer to such a node x as is described in Condition (6).

We show $\Vdash$ "$\check{T}$ is not R-embeddable". Suppose, to the contrary, that $p_0 \Vdash$ "$\dot{f} : \check{T} \to \mathbf{R}$ is an order preserving map". Put $F = \{\langle x, a, p\rangle \in T \times \mathbf{Q} \times P \mid p \Vdash \check{a} \leq \dot{f}(\check{x})\}$. Then the set $S = \{\lambda \in \omega_1 \cap \mathrm{Lim} \mid F \cap (T \mid \lambda \times \mathbf{Q} \times P \mid \lambda) = \phi_\lambda\}$ is stationary. Consider the club set C that consists of the ordinals $\lambda \in \omega_1$ satisfying the following condition:

(i) $\forall x \in T \mid \lambda \forall p \in P \mid \lambda \forall a \in \mathbf{Q}^+ (\exists \langle q, y\rangle \in P \times T (q \supseteq p \wedge y <_T x \wedge q \Vdash \check{a} < \dot{f}(\check{y})) \to$
$\exists$such $\langle q, y\rangle$ in $P \mid \lambda \times T \mid \lambda)$.

Suppose $\lambda \in C \cap S$. Then, it is easy to see that $\phi_\lambda \neq \emptyset$. So T_λ contains the extra point w. Let v be its extension in $T_{\lambda+1}$, and take $b \in \mathbf{Q}$ and $p_1 \in P$ such that $p_1 \supset p_0$ and $p_1 \Vdash \dot{f}(\check{w}) < \check{b} < \dot{f}(\check{v})$. Let $x \in T \mid \lambda$ be a $(p_1 \mid \lambda, b)$-search point for w. Then there are two cases.

First case. $\langle x, b, p_2\rangle \in \phi_\lambda$ for some $p_2 \in P \mid \lambda$ with $p_2 \supset p_1 \mid \lambda$. Then $p_2 \cup p_1 \Vdash \dot{f}(\check{w}) < \check{b} <$

$\dot{f}(\check{x})$, which contradicts $p_0 \Vdash \dot{f}$ is order preserving.

Second case, $\forall p_2 \in P \mid \lambda \forall z \in T \mid \lambda (p_2 \supset p_1 \mid \lambda \wedge z >_T x \to \langle z, b, p_2 \rangle \notin \phi_\lambda)$. This means, by $\lambda \in C \cap S$, $\forall p_2 \in P \forall z \in T (p_2 \supset p_1 \mid \lambda \wedge z >_T x \to$ not $p_2 \Vdash \check{b} \leq \dot{f}(\check{z}))$. But this contradicts $p_1 \Vdash \check{b} < \dot{f}(\check{v})$. The lemma is thus proved.

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