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# On Elementary Deformations of Regular Homotopies II

by

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## § 1. Introduction

We work in the piecewise linear category.

Homma and the author [HN-2] have shown that any homotopy between two nice maps of a closed surface into a 3-manifold is decomposed into a finite sequence of six kinds of local moves and their inverses, called *elementary deformations*. Among the elementary deformations, two kinds of deformation and their inverses are moves in a neighborhood of branch points. They do not occur in regular homotopies of immersions. Here we are interested in only other four kinds of deformation shown in Figure 1.1. The inverse deformations of an elementary deformation of type $i$ ($i=1, 2, 3$) is said to be of type $i^-$. Also an elementary deformation of type $i$ is often said to be of type $i^+$.

In [Ng] we have proved:

*Theorem 1.1. Let $f, g : F \to M$ be nice immersions of a surface to the interior of a 3-manifold. If $f$ and $g$ differ only on a disk in $F$ such that $f$ and $g$ send the disk into a 3-ball in $M$, then $f$ is deformed to $g$ by a finite sequence of elementary deformations of type $1^\pm$, $2^\pm$, $3^\pm$, and 6.*

In this note we first establish a criterion to determin if two given nice immersions of a surface into a closed oriented 3-manifold are regularly homotopic via the framed link theory (Theorem 3.1), and then we show that any regular homotopy between nice immersions of a surface into a closed oriented 3-manifold is decomposed into the four kinds of elementary deformations and their inverses (Corollary 3.5). One of the tools in the proof of Theorem 1.1 is SM-moves which are compositions of elementary deformations of type $1^\pm, 2^\pm, 3^\pm$, and 6 (see [Ng] for the definition of SM-moves). In the proof of Theorem 3.1, SM-moves are used to rewind the twist of a neighborhood of an essential loop in an annulus.

Throughout this note we use the terminology for framed links in [Kp]

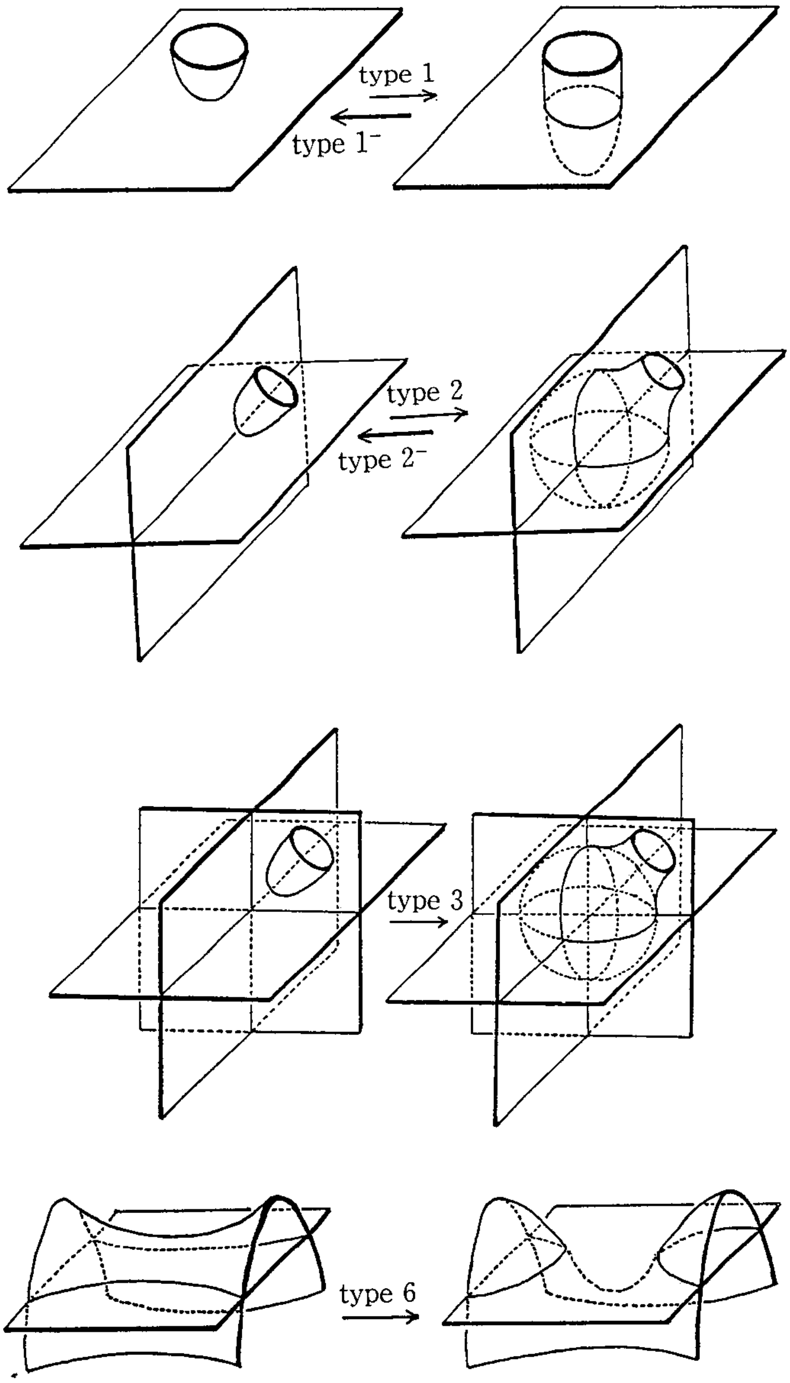



Fig. 1. 1

without notice. And we also assume that $F$ is a surface and $M$ is a closed oriented 3-manifold.

## § 2. A twisting number

In this section we define a twisting number for annuli and Moebious bands in an oriented closed 3–manifold $M_L$ with even framings.

Any closed connected, oriented 3–manifold $M$ is $M_L$ for some framed link $L \subset S^3$ [Lc]. Further the link $L$ can be chosen to be a framed link with even framings [Kp]. Furthermore each component of the link $L$ is a trivial knot in $S^3$ [BP]. Hence from now on, a framed link is assumed to be a framed link with *even* framing, each of whose components is a *trivial* knot in $S^3$.

Let $K$ be a Moebious band in $S^3$. We define a twisting number $Tw(K)$ of $K$ as follows: Let $l$ be a "central" loop in $K$. Orientate $l$ and $K$ such that 1–dimensional homology class $[\partial K]$ of $H_1(K)$, represented by $\partial K$, is twice of $[l]$. Define

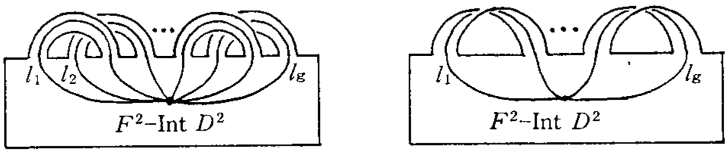
$Tw(K) = [lk(l, \partial K/2]$, where $[\ldots]$ means Gauss' symbol.

Note that $Tw(K)$ is equal to the twisting number of an annulus obtained by twisting $K$ by a half in the left hand sense.

For a closed surface $F$, let

$\Omega = \{l_1, \ldots, l_g\}$

be a standard system of loops on the surface $F$, as shown in figure 2.1 such that




$D^2$: a disk in $F^2$

**Fig. 2.1**

(1) $F - \cup l_i$ is an open 2–cell,

(2) $\{[l_i] \mid i=1, \ldots, g\}$ is a generating set of $H_1(F)$, and

(3) if $F$ is non–orientable, then a regular neighborhood of $l_i$ $(i=1, \ldots, g)$ is a Moebious band.

Let $f : F \to M$ be a nice immersion of the surface to a closed connected oriented 3-manifold which embeds the union of the loop $l_i$. For any framed link $L$ with $M_L = M$, we define an element $N_L(f) \in \mathrm{Hom}(H_1(F), Z/2Z)$ as follows: Deforming $f(F)$ by an ambient isotopy of $M$, if necessary, we can assume that the map $f$ embeds a small regular neighborhood $N_i$ of each loop $l_i$ into $S^3 - L$. Considering $S^3 - L \subset S^3$, we may assume that $f(N_i)$ lies in the 3-sphere $S^3$. Hence the twisting number of $f(N_i)$ in $S^3$ is defined, say $n_i$. Set

$N_L(f)\ ([l_i]) = n_i \bmod 2.$

Since $\{[l_i] \mid i=1,\ldots,g\}$ is a generating set of $H_1(F)$, $N_L(f)$ can be extended to a homomorphism of $H_1(F)$ to $Z/2Z$ by the natural way.

*Lemma 2.1. $N_L(f)$ is well defined.*

**Proof.** We must show that $N_L$ is independent from the choice of an ambient isotopy of $M$ in the definition. Suppose that $f(N_i)$ crosses over a component $k$ of $L$ by an ambient isotopy of $M$ (see Figure 2.2). If the framing number of the knot $k$ is $2m$, then $f(N_i)$ is deformed to an annulus or a Moebious band $A_i$ which is almost parallel to $f(N_i)$ and coils round $2m$ times around the knot $k$ (see Figure 2.2). This deformation $\Gamma$ is decomposed to two kinds of deformations: one is a deformation $\Gamma_1$ which makes $f(N_1)$ $2m$ times full-twisted, while the other is a deformation $\Gamma_2$ which exchange parts of $f(N_i)$. Hence the result follows from the following lemma by the note for the definition of the twisting number for a Moebious band in $S^3$. ∎

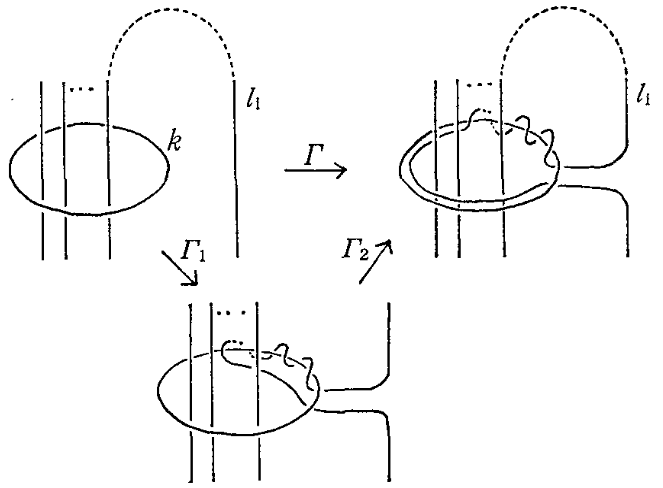



Fig. 2.2

*Lemma 2.2. Let $N$ be an annulus in the oriented 3-sphere $S^3$. By exchanging a part of the annulus as shown in Figure 2.3, the twisting number of the annulus changes by $\pm 2$.*

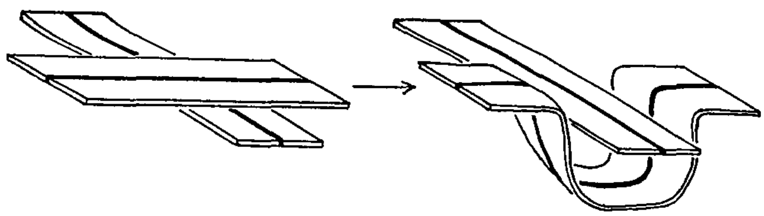


Fig. 2.3

**Proof.** Since the twisting number is defined by the linking number of the boundary curves of the annulus and the linking number is bilinear, it is sufficient to examine the case that the annulus is unknotted, i.e. lies on the disk. But this case is obvious. ▩

*Lemma 2.3.* *$N_L(f)$ is invariant by regular homotopies. Hence it is also invariant by elementary deformations of type $1^{\pm}$, $2^{\pm}$, $3^{\pm}$, and 6.*

**Proof.** When the loop $l_i$ crosses itself by a regular homotopy the twisting number of $f(N_i)$ changes by $\pm 2$ by the above lemma, where $N_i$ is a regular neighborhood of $l_i$ in $F$. Since the range of $N_L(f)$ is $Z/2Z$, the result follows. ▩

## § 3. Main theorem

In this section we prove:

*Theorem 3.1.* *Let $f_1$, $f_2 : F \to M$ be nice immersions which are homotopic. Then $f_1$ deforms to $f_2$ by a finite sequence of elementary deformations of type $1^{\pm}$, $2^{\pm}$, $3^{\pm}$, and 6, if and only if $N_L(f_1)=N_L(f_2)$ for any framed link $L$ with $M_L=M$.*

We needs the following three lemmas.

*Lemma 3.2.* *Let $f : F \to M$ be a continuous map which is an immersion on the outside of a disk $D$ in $F$. Then for any $\varepsilon>0$, there exists a nice immersion $f'$ with $d(f,f')<\varepsilon$ such that the maps $f$ and $f'$ coincide on the outside of the disk $D$.*

**Proof.** This can be shown by a parallel arguments with those of Lemma 3.2 in [HN-1]. ▩

Modification of a nice map $f : F \to M$ to a new nice map $g : F \to M$ is called an *h-move* if there exist a disk $D$ in $F$ and a 3-ball $B$ in $M$ such that the images $f(D)$ and $g(D)$ are contained in the interior of the 3-ball $B$ and that the maps $f$ and $g$ coincide outside the disk $D$. The pair $(D, B)$ is called a *support pair* of the $h$-move.

*Lemma 3.3.* *Let $f, g : F \to M$ be nice maps which are homotopic. Then $f$ deforms to $g$ by a finite sequence of $h$-moves.*

**Proof.** This lemma is same with Theorem in §3 in [HN-1]. ▩

*Lemma 3.4.* *Let $f, g : F \to M$ be nice maps which are homotopic. Then for any $\varepsilon>0$, there exists a finite sequence of nice immersions $f=g_0, \dots, g_n$ : $F \to M$ such that each $g_i$ is deformed to $g_{i+1}$ by a single $h$-move, and that*

$d(g_n, g) < \varepsilon$.

**Proof.** Let $f=f_0, f_1, \ldots, f_n=g$ be a sequence of nice maps, assured by Lemma 3.3, such that each $f_i$ is deformed to $f_{i+1}$ by a single $h$-move with a support pair $(D_i, B_i)$. Let $\delta$ be a positive real number such that

(1) $\delta < \varepsilon$ and
(2) $\delta < \min\{d(f_i(D_i) \cup f_{i+1}(D_i), M-B_i) \mid i=0, \ldots, n-1\}$.

Suppose that $g_i$ is defined and $d(f_i, g_i) < \delta \times i/n$. Then we define $g_{i+1}$ as follows: Since $f_i(D_i)$ lies in the ball $B_i$, by the general position argument, there exists a nice map $\tilde{g}_{i+1}$ such that $\tilde{g}_{i+1}$ and $g_i$ coincide on the outside of the disk $D_i$ and that $d(\tilde{g}_{i+1}, g_i) < \delta \times i/n + \delta/2n$. By the help of Lemma 3.2, on the disk $D_i$, we can deform $\tilde{g}_{i+1}$ to a nice immersion $g_{i+1}$ with $d(\tilde{g}_{i+1}, g_{i+1}) < \delta/2n$ such that $\tilde{g}_{i+1}$ and $g_{i+1}$ coincide the outside of the disk $D_i$. Since $g_{i+1}(D_i)$ still lies in the ball $B_i$, the map $g_i$ is deformed to $g_{i+1}$ by a single $h$-move with a support pair $(D_i, B_i)$. This completes the proof. ∎

**Proof of Theorem 3.1.** If $f_1$ is deformed to $f_2$ by a finite sequence of elementary deformations, they are regularly homotopic. Hence the necessity follows from Lemma 2.3.

For sufficiency, suppose that $N_L(f_1) = N_L(f_2)$. If the surface $F$ is the 2-sphere, then let $l_0$ be the equator of the 2-sphere, else let $l_0$ be the empty set. By Lemma 3.4 and the main theorem in [Ng], there exists a finite sequence of elementary deformations of type $1^{\pm}$, $2^{\pm}$, $3^{\pm}$, and 6, by which the map $f_1$ is deformed to a nice immersion $f_1'$ sufficiently close to the map $f_2$. By the necessity which we have proved now, we have $N_L(f_1') = N_L(f_1)$. Thus we have $N_L(f_1') = N_L(f_2)$. Now we may assume that each $f_1'(l_i)$ lies in a regular neighborhood $V_i$ of $f_2(l_i)$. Since the two maps are very close, the each loop $f_1'(l_i)$ goes around inside the solid torus $V_i$ exactly once. By applying elementary deformations of type 1, if necessary, we can assume that $f_1'(l_i)$ is parallel to $f_2(l_i)$ in $V_i$. Hence applying elementary deformations, we can assume that $f_1'$ and $f_2$ coincide on each loop $l_i$.

Let $N_i$ be a small regular neighborhood of $l_i$ in $F$. Since $N_L(f_1') = N_L(f_2)$, the difference of the two twisting numbers of $f_1'(N_i)$ and $f_2(N_i)$ is even. This is also true for $N_0$ by Lemma 4.4 in [Ng]. By applying $SM$-moves and elementary deformations in a small area suitably many times (cf. Lemma 2.5 in [Ng]), we can further assume that the twisting numbers of $f_1'(N_i)$ and $f_2(N_i)$ are same. Hence we can assume that the map $f_1'$ and the map $f_2$ coincide on $N_i$.

Since the map $f_2$ is an immersion, there exists an immersion $\Phi : F \times [-1, 1] \to M$ such that

(1) $\Phi(x, 0) = f_2(x)$ for all $x \in F$, and
(2) for each $x \in F$, there exists a neighborhood $U$ of $x$ in $F$, so that the

restriction $\Phi \mid U\times I$ is an embedding.

Let $\phi_2 : F\to F\times[-1,1]$ be the map defined by $\phi_2(x)=(x,0)$ for all $x\in F$. Since the two maps $f_1'$ and $f_2$ are very close, there exists a unique nice immersion $\phi_1 : F\to F\times[-1,1]$ such that

(1) $\Phi\circ\phi_1=f_1'$, and
(2) the maps $\phi_1$ and $\phi_2$ coincide on a neighborhood of the set $\cup\{l_i \mid i=0,\dots,n\}$ in the surface $F$.

These are two cases.

**Case 1.** The surface $F$ is the 2-sphere: The maps $f_1'$ and $f_2$ coincide on a neighborhood of the equator. Note that $\Phi\circ\phi_2=f_2$ and that the maps $\phi_1$ and $\phi_2$ are very close. Let $H_+$ and $H_-$ be the northern and southern hemispheres of $F$. Since $f_1'$ and $f_2$ are very close, the images $\phi_1(H_+)$ and $\phi_2(H_+)$ lie in a 3-ball in $F\times[-1,1]$. Now apply Corollary 4.9 in [Ng] on the disk $H_+$ so that we can make the maps $\phi_1$ and $\phi_2$ coincide on the northern hemisphere. By the similar way we can also make $\phi_1$ and $\phi_2$ coincide on the southern hemisphere, too. The deformation of $\phi_1$ to $\phi_2$ induces a finite sequence of elementary deformations which converts $f_1'$ to $f_2$.

**Case 2.** $F$ is not 2-sphere: Let $p : G\to F$ be the universal covering and $p\times id$: $G\times I\to F\times I$ be the map defined by $p\times id(g,t)=(p(g),t)$ for all $g\in G$ and $t\in I$. Then $p\times id : G\times I\to F\times I$ is a covering map. Let $\tilde{\phi}_2 : G\to G\times I$ be the map defined by $\tilde{\phi}_2(g)=(g,0)$ for all $g\in G$. Since the two maps $\phi_1$ and $\phi_2$ are homotopic, there is a unique immersion $\tilde{\phi}_1 : G\to G\times[-1,1]$ such that

(1) $(p\times id)\circ\tilde{\phi}_1=\phi_1$, and
(2) for a neighborhood $W$ of the set $\cup\{l_i \mid i=1,\dots,n\}$ in $F$, the maps $\tilde{\phi}_1$ and $\tilde{\phi}_2$ coincide on the set $p^{-1}(W)$.

Note that $\tilde{\phi}_1$ is homotopic to the map $\tilde{\phi}_2$. Let $D$ be a disk in $F-\cup\{l_i \mid i=1,\dots,n\}$ such that the two maps $f_1'$ and $f_2$ coincide on the outside of the disk. Let $\tilde{D}$ be a component of $p^{-1}(D)$. For a sufficiently big disk $D_*$ in $G$, the images $\tilde{\phi}_1(\tilde{D})$ and $\tilde{\phi}_2(\tilde{D})$ lie in the 3-ball $D_*\times I$. Hence we can assume that, by a finite sequence of $h$-moves, the map $\tilde{\phi}_1\mid\tilde{D}$ deforms to the map $\tilde{\phi}_2\mid\tilde{D}$ keeping the boundary collar of the disk $\tilde{D}$ fixed. Further we can assume that the support pairs of the $h$-moves are very small so that each elementary deformations induces an elementary deformation in $F\times I$ (and hence an elementary deformation in $M$ by Condition (2) for $\Phi$). Thus this induces a finite sequence of $h$-moves by which the map $f_1'$ deforms to the map $f_2$. Therefore the result follows from Theorem 1.1. ∎

*Corollary 3.5. Let $f_1, f_2 : F\to M$ be nice immersions of a closed surface*

*$F$ to a closed, connected, oriented 3–manifold $M$. Then $f_1$ and $f_2$ are regularly homotopic if and only if one is deformed to the other by a finite sequence of elementary deformations of type $1^{\pm}$, $2^{\pm}$, $3^{\pm}$, and 6.*

**Proof.** Note that an elementary deformation of type $3^-$ is a composition of elementary deformations of other types. Hence the result follows immediately from the above theorem because regular homotopies do not change the element $N_L(f_i)$. ▩

*Note*: Morin and Petit [MP] showed that only six types of modifications are needed for eversion of the 2–shpere. Their six deformations are compositions of our deformations and vice versa.

## § 4. An application

Let $F$ be a surface with boundary. If $F$ is orientable, then $F$ is a connected sum of tori and a 2–sphere with holes. If $F$ is non–orientable, then $F$ is a connected sum of projective spaces and a 2–sphere with holes. Let $\Omega=\{l_1, \dots, l_g\}$ be a 'standard' system of loops on $F$ as shown in Figure 4.1. Now for each immersion $f: F \to M$, we can define $N_L(f)$ by using the system $\Omega$. Let $f_1, f_2: F \to M$ be immersions of a surface (possibly with boundary) to a closed, connected, oriented 3–manifold $M$. Suppose that they are homotopic. Define

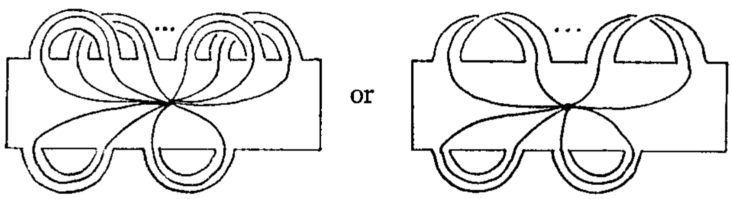


Fig. 4.1

$N(f_1, f_2) \in \mathrm{Hom}(H_1(F), Z/2Z)$ by

$$N(f_1, f_2) = N_L(f_1) - N_L(f_2) \pmod 2,$$

where $L$ is a framed link with $M_L = M$.
By Theorem 3.1, $N(f_1, f_2)$ is independent from the choice of the framed link $L$, if $F$ is closed. Even if $F$ is not closed, well–definedness of $N(f_1, f_2)$ has been proved in the proof of Theorem 3.1.

*Lemma 4.1. Let $f_1, f_2: F \to M$ be immersions of a surface (possibly with boundary) to a closed, connected, oriented 3–manifold $M$. Suppose that they are homotopic. Then the two maps $f_1$ and $f_2$ are regularly homotopic if and only if $N(f_1, f_2)=0$.*

**Proof.** If $F$ is closed, then the result follows from Theorem 3.1. Suppose that $F$ possesses a boundary. Necessity follows from Lemma 2.3. For sufficiency, suppose that $N(f_1, f_2)=0$. Note that any immersions of $F$ are regularly homotopic, if they coincide on a regular neighborhood of the standard system $\Omega$ on $F$. Since they are homotopic, we can assume that the two immersions coincide on $\Omega$ via a regular homotopy. Since $N_L(f_1)=N_L(f_2)$, the result follows from the process of making maps coincide on a regular neighborhood of the standard system of loops in Theorem 3.1. ∎

The following theorem is a generalization of Corollary 1.4 in [JT] (cf. [Hr] and [Sm]).

*Theorem 4.2. The regular homotopy classes of immersions of a surface $F$ (possibly with boundary) into a closed, connected, oriented 3-manifold $M$ are in one-one correspondence with $Hom(H_1(F), Z/2Z) \times [F, M]$, where $[F, M]$ is the homotopy classes of maps of $F$ to $M$.*

**Proof.** It is sufficient to show that for each homotopy class of a map $h: F \to M$, the number of regular homotopy classes in the homotopy class of the map $h$ is equal to that of elements in $\mathrm{Hom}(H_1(F), Z/2Z)$ because the map $h$ is homotopic to a nice immersion, say $f$. Let $\Omega=\{l_1, \ldots, l_g\}$ be a standard system of loops on $F$. There are two cases:

**Case 1.** $\partial F \neq \phi$: There exists a system of mutually disjoint proper arcs $\{l_1^*, \ldots, l_g^*\}$ on $F$ such that

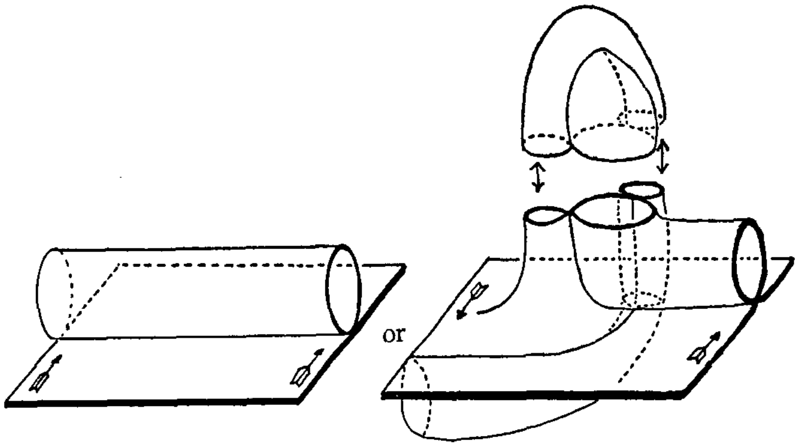
(1) $l_i^* \cap l_j = \phi$ if $i \neq j$,
(2) $l_i^* \cap l_j$ is one crossing point, and
(3) $f \mid l_i^*$ is an embedding.

Let $N_i^*$ be a regular neighborhood of $l_i^*$ in $F$. We can assume that $f \mid N_i^*$ is still an embedding. Let $f_i^*$ be a map obtained by full-twisting $f(F)$ on $f(N_i^*)$. Then

$$N(f_i^*, f)\ ([l_i])=\delta_{ij}.$$

Since the above modification can be performed individually, the result follows.

**Case 2.** $\partial F=\phi$: The standard system $\Omega$ on $F$ is a standard system of $F-\mathrm{Int}\ D$, where $D$ is a disk on $F$. Hence there exists a system of loops $\{l_1^\#, \ldots, l_g^\#\}$ such that $\{l_i^\# \cap (F-\mathrm{Int}\ D) \mid i=1, \ldots, g\}$ play the role of $\{l_i^*\}$ of Case 1. Let $N_i^\#$ be a regular neighborhood of $l_i^\#$ in $F$. We can assume that $f \mid N_i^\#$ is an embedding. If $N_i^\#$ is an annulus, then modify the map $f$ on $N_i^\#$ to a map $f_1^\#$ as shown in the left picture in Figure 4.2. If $N_i^\#$ is a Moeboius band, then modify the map $f$ on $N_i^\#$ to a map $f_i^\#$ as shown in the right picture in Figure 4.2. Then




glue the laterals along arrows to get an immersed annulus

glue the laterals along arrows to get an immersed Moebious band

Fig. 4. 2

$N(f_i^{\#}, f)\ ([l_i]) = \delta_{ij}$.

Again the above modification can be performed individually, the result follows. ∎

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