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Review of the $N=2$ Superconformal Field Theory

by

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Abstract

The classification of the $N=2$ superconformal field theory is reviewed. The Kac determinant formula is used to determine all the unitary representations. The characters are expressed by the string functions and the theta functions of the $SU(2)$ Kac-Moody algebra and their modular transformations are shown. The classification of the modular invariant partition functions of the minimal models are represented. Finally an application to the string compactification by tensoring the minimal theories is briefly reviewed.

Contents

§ 1. Introduction

Since the proposal of the conformal field theory [1], the extensive investigations were accomplished. The unitary representation of the Virasoro algebra, which is a central extention of the commutation relation of the stress-energy tensor were determined by Friedan-Qiu-Shenker [2] and reconstructed by the coset space construction for $0<c<1$ (where c is the value of the central charge) by Goddard-Kent-

Olive [3]. Cardy has observed that it is important to study the modular invariance of the partition function on the torus to classify the conformal field theories [4]. The method is also applicable to the extended algebras, among which the representation theory of the Wess-Zumino-Witten models were investigated by Gepner-Witten [5]. The complete classifications of the $SU(2)$ Kac-Moody algebra and of the Virasoro algebra for the $0<c<1$ cases were realized in refs. [6, 7, 8] recently, which correspond to the classification of Lie algebras $A-D-E$. For the $c=1$ case, the theory is that of a free boson compactified on a circle, $\phi: S^1 \to C$, or an orbifold S^1/Z_2. The plausibly complete classification of this case were now available [9].

The $N=1$ superconformal algebra has been also extensively studied. The representations were determined in ref. [10] and the character formula were derived in ref. [11] for the case of $0<c<\frac{3}{2}$. The $A-D-E$ classification was almost completed by A. Cappell [12]. For the $c=\frac{3}{2}$ case almost complete classifications are presented [13] by means of the same method as ref. [9].

In this paper we concentrate on the review of the $N=2$ superconformal field theory. $N=2$ superconformal algebra has applications to critical statistical systems in two-dimensions. It also arises in the compactifications of heterotic superstring. Knowledge about the unitary representations of the $N=2$ superconformal algebra provides us with some nonperturbative tools that may be used to gain insight in the physics.

The superconformal transformation in $N=2$ superspace (z, θ^1, θ^2) is defined by

$$\delta z = \varepsilon(z) - \theta^i \eta^i(z), \tag{1.1a}$$

$$\delta\theta^i = \frac{1}{2}\theta^i \partial_z \varepsilon(z) + \eta^i(z) + i\partial_z \eta^j(z)\theta^j\theta^i + \varepsilon^{ij}\theta^j \xi(z), \tag{1.1b}$$

where θ^i is the Grassmann variable and the conformal, supergauge and conformal $O(2)$ transformations are parametrized by $\varepsilon(z)$, $\eta^i(z)$ and $\xi(z)$, respectively. It is convenient to use radial variable z in stead of τ, σ which characterize the two-dimensional world sheet. The mapping is given as follows

$$(\ln z, z^{-\frac{1}{2}}\theta^1, z^{-\frac{1}{2}}\theta^2) \longleftrightarrow (\tau + i\sigma, \theta^1, \theta^2), \tag{1.2}$$

where τ is a time coordinate and σ is a space one $(0<\sigma\leq\pi)$. $z=\exp(\tau+i\sigma)$ is called the cylinder variable. We can obtain $N=2$ superconformal algebra from the transformation (1.1) (see § 2).

The importance of the $N=2$ supersymmetry on the world sheet was pointed out in the context of the string compactifications, in which four-dimensional space-time supersymmetry is preserved [31]. The most significant application of the $N=2$ superconformal structure to the compactifications of the heterotic string will be discussed in § 8.

In § 3, § 4, § 5 we described the unitary representations and discussed the character formula. Kac-determinant formula for the $N=2$ algebra was derived and the whole unitary representations are determined in ref. [14]. It plays an important role of unitary representations. The explicit unitarity representations for the case $1 \leq c < 3$ are constructed by means of the coset construction $SU(2) \times U(1)/U(1)$ in ref. [15]. The character formula were derived explicitly in ref. [16] by means of Rocha-Caridi method [17]. But the formula seen to be too complicated to apply them to study the properties in modular transformations. The character formula in terms of the theta functions and string functions of $SU(2)$ Kac-Moody algebra has been derived more physically by using the parafermions originally proposed by Zamolodchikov and Fateev [18].

In § 6, § 7 we classified the modular invariant partition functions. Modular invariant quantities are constructed from the characters of the representation involved. In fact, the partition function is written in terms of the characters with the non-negative integer multiplicities. The structure of the multiplicities agree with the structure of Lie algebras $A-D-E$. Gepner and Qiu have constructed the modular invariant partition functions and have completed the $A-D-E$ classification of Z_k parafermionic theories. By applying the parafermion theory, the minimal theories of the $N=2$ superconformal algebra were also completely classified in ref. [20] for the A and P algebra. For the case of T algebra, though the extensive investigations are available, but we only mention on the relations between the twisted $N=2$ algebra and the twisted $SU(2)$ Kac-Moody algebra recently observed in refs. [22, 23].

There are many completely solvable, extended conformal field theories, not mentioned in this article [23, 24, 25, 26, 27]. For the review of the present status on the classifications of conformal field theories, see refs. [29, 30].

We will shortly review on the method to construct the exactly solvable string compactifications proposed by Gepner [20, 32]. In appendix A, we have demonstrated the derivations of Kac determinant formula by Kato and Matsuda [14]. In appendix B the parafermionic construction of character formula following Qiu and Gepner [20].

§ 2. The $N=2$ Superconformal Algebra

The $N=2$ superconformal algebra is generated by the stress-energy tensor $T(z)$, two super partners $G^i(z)$, $(i=1,2)$ and $U(1)$ Kac-Moody algebra $J(z)$ with scale dimensions $h=2, 3/2, 3/2$ and 1 respectively. Their operator product expansions (OPF) are:

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \text{reg.}, \qquad (2.1\text{a})$$

$$T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{J'(w)}{z-w} + \text{reg.}, \qquad (2.1\text{b})$$

$$J(z)J(w)\sim\frac{\frac{c}{3}}{(z-w)^2}+\text{reg.},\tag{2.1 c}$$

$$T(z)G^i(w)\sim\frac{\frac{3G^i(w)}{2}}{(z-w)^2}+\frac{G^i(w)'}{z-w}+\text{reg.},\tag{2.1 d}$$

$$J(z)G^i(w)\sim i\varepsilon^{ij}\frac{G^j(w)}{z-w}+\text{reg.},\tag{2.1 e}$$

$$G^i(z)G^j(w)\sim\delta^{ij}\frac{\frac{2}{3}c}{(z-w)^3}+\delta^{ij}\frac{2T(w)}{z-w}+i\varepsilon^{ij}\Big[\frac{2J(w)}{(z-w)^2}+\frac{J'(w)}{z-w}\Big]+\text{reg.}\tag{2.1 f}$$

$T(z), G^i(z)$ and $J(z)$ are Laurant expanded as:

$$T(z)=\sum_n L_n z^{-n-2},\tag{2.2 a}$$

$$G^j(z)=\sum_n G^i_n z^{-n-\frac{3}{2}},\tag{2.2 b}$$

$$J(z)=\sum_n J_n z^{-n-1}.\tag{2.2 c}$$

And the $N=2$ algebra is derived by inserting (2.2) into (2.1):

$$[L_m, L_n]=(m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n,0},\tag{2.3 a}$$

$$[L_m, J_n]=-nJ_{m+n},\tag{2.3 b}$$

$$[J_m, J_n]=\frac{c}{3}m\delta_{m+n,0},\tag{2.3 c}$$

$$[L_m, G^i_n]=(\frac{m}{2}-n)G^i_{m+n},\tag{2.3 d}$$

$$[J_m, G^i_n]=i\varepsilon^{ij}G^j_{m+n},\tag{2.3 e}$$

$$\{G^i_m, G^j_n\}=2\delta^{ij}L_{m+n}+i\varepsilon^{ij}(m-n)J_{m+n}+\frac{c}{3}(m^2-\frac{1}{4})\delta^{ij}\delta_{m+n,0}.\tag{2.3 f}$$

All of the generators satisfy the hermiticity conditions of the form:

$$A^\dagger_m=A_{-m},\tag{2.4}$$

which follow from the reality of the super stress-energy tensor.

The three types of the $N=2$ superalgebra are given by three modings of the supersymmetry generators, corresponding to the three ways of choosing boundary conditions on the cylinder, i. e.,

A sector:

$$G^i(ze^{2\pi i})=-G^i(z), (i=1, 2),$$

P sector:

$$G^{i}(ze^{2\pi i})=G^{i}(z),\ (i=1,2),$$

T sector:

$$\begin{aligned}G^{1}(ze^{2\pi i})&=\pm G^{1}(z),\\ G^{2}(ze^{2\pi i})&=\mp G^{2}(z).\end{aligned} \tag{2.5}$$

The L_m's always integrally moded, because the bosonic stress-energy tensor is always periodic on the cylinder. The A (for anti-periodic) algebra has integer modes for J_m, but half integer modes for G^i_m. The P (for periodic) algebra has integer modes for J_m and G^i_m. The T (for twisted) algebra has integer modes for G^1_m, but half integer modes for J_m and G^2_m. Furthermore P sector splits into two subsectors $P^{\pm}$, by the specification of the vacuum. The zero modes are L_0, J_0 and G_0 in the P algebra. There are two kinds of the highest weight vector (hwv) $|h, q\mp\frac{1}{2}\rangle$ with energy $L_0=h$ and charge $J_0=q\mp\frac{1}{2}$. Each satisfies an additional highest weight condition with respect to the charge,

$$(G^1_0\pm iG^2_0)\,|h, q\mp\frac{1}{2}\rangle\pm=0. \tag{2.6}$$

These two representations $P^{\pm}$ are isomorphic under conjugation,

$$J_m\rightarrow -J_m,\ G^2_m\rightarrow -G^2_m. \tag{2.7}$$

For the A algebra the zero modes are L_0 and J_0, so a hwv $|h, q\rangle$ is characterized by its energy (L_0 eigenvalue) h and charge (J_0 eigenvalue) q.

Using these charges we can define the character $\chi^{A,P\pm}$ on the torus as:

$$\chi^{A,P\pm}(\tau, z)=Tr(q^{L_0-\frac{c}{24}}y^{J_0}), \tag{2.8}$$

where $q=\exp(2\pi i\tau)$, $y=\exp(2\pi iz)$, with $\tau, z\in \boldsymbol{C}$; $Im\ \tau>0$ (τ is the modular parameter for the torus and z measures the $U(1)$ charge) and the universal Casimir energy factor $q^{-\frac{c}{24}}$ is included [4].

For the T algebra the zero modes are L_0 and G^1_0. A hwv $|h\rangle$ is characterized by its energy h. Each level n (see the next section) splits into two equal subspaces of fermion parity $(-1)^F=\pm 1$, where $(-1)^F$ is the operator which commutes with L_m and J_m, but anticommutes with G^i_m and

$$(-1)^F|h\rangle=|h\rangle. \tag{2.9}$$

The character is defined as:

$$\chi^{T}(\tau)=Tr(q^{L_0-\frac{c}{24}}). \tag{2.10}$$

§ 3. The Unitary Representations

When one determines the unitary representations of the $N=2$ super algebras and calculates the characters, the Kac determinant formula plays essential roles to treat the null states of the Verma module. They are polynomial expressions in c, h and q for the determinant of the matrices of the inner products of a basis of ordered monomials for given eigenspace (up to a basis dependent positive constant). The vanishing surfaces, or curves of the determinant formula describe the Verma modules which contain null vectors. The Verma modules of all three algebras can be decomposed into eigenspaces of L_0, called levels. Level n is the eigenspace with L_0 eigenvalue $h+n$.

For the A algebra each level of the Verma module can be further decomposed into J_0 eigenspaces with eigenvalue $q+m$, where m is called the relative charge. The counting of states is summarized by the partition function $P_A(n,m)$ defind by

$$\sum_{n,m} P_A(n,m)x^n y^m = \prod_{k=1}^{\infty} \frac{(1+x^{k-\frac{1}{2}}y)(1+x^{k-\frac{1}{2}}y^{-1})}{(1-x^k)^2}. \tag{3.1}$$

For the P algebra, again, each level n is decomposed by relative charge m, with partition function $P_P(n,m)$

$$\sum_{n,m} P_P(n,m)x^n y^m = (y^{\frac{1}{2}}+y^{-\frac{1}{2}}) \prod_{k=1}^{\infty} \frac{(1+x^k y)(1+x^k y^{-1})}{(1-x^k)^2}. \tag{3.2}$$

We also need partition functions $\tilde{P}_{A,P}$ in the presence of single charged fermions:

$$\sum_{n,m} \tilde{P}_{A,P}(n,m\,;k)x^n y^m = (1+x^{|k|}y^{\mathrm{sgn}(k)})^{-1} \sum_{n,m} P_{A,P}(n,m)x^n y^m. \tag{3.3}$$

where $\mathrm{sgn}(k)=1$ for $k>0$, $\mathrm{sgn}(k)=-1$ for $k<0$ and $\mathrm{sgn}(0)=\pm 1$ for the representation $P^{\pm}$.

For the T algebra, the partition function P_T for each fermion parity is

$$\sum_{n} P_T(n)x^n = \prod_{k=1}^{\infty} \frac{(1+x^k)(1+x^{k-\frac{1}{2}})}{(1-x^k)(1-x^{k-\frac{1}{2}})}. \tag{3.4}$$

For the A and P algebras, let $M^P_{n,m}$ and $M^A_{n,m}$ be the inner product matrices for level n and relative charge m. The Kac determinant formula for the A algebra is:

$$\det M^A_{n,m}(c,h,q) = \prod_{1\leq rs\leq 2n,\, s;\,\mathrm{even}} (f^A_{r,s})^{P_A(n-\frac{rs}{2},\, m)} \prod_{k\in \mathbf{Z}+\frac{1}{2}} (g^A_k)^{\tilde{P}_A(n-|k|,\, m-\mathrm{sgn}(k)\,;k)} \tag{3.5}$$

where

$$f^A_{r,s}(c,h,q) = 2(\frac{c}{3}-1)h - q^2 - \frac{1}{4}(\frac{c}{3}-1)^2 + \frac{1}{4}[(\frac{c}{3}-1)r+s]^2,\ (s\,;\ \mathrm{even}), \tag{3.6a}$$

$$g^A_k(c,h,q) = 2h - 2kq + (\frac{c}{3}-1)(k^2-\frac{1}{4}),\ (k\in \mathbf{Z}+\frac{1}{2}). \tag{3.6b}$$

For the P algebra,

$$\det M^P_{n,m}(c,h,q) = \prod_{1\leq rs\leq 2n,\, s;\,\mathrm{even}} (f^P_{r,s})^{P_P(n-\frac{rs}{2},\, m)} \prod_{k\in \mathbf{Z}} (g^P_k)^{\tilde{P}_P(n-|k|,\, m-\mathrm{sgn}(k)\,;k)} \tag{3.7}$$

where

$$f^P_{r,s}(c,h,q)=2(\frac{c}{3}-1)(h-\frac{c}{24})-q^2+\frac{1}{4}[(\frac{c}{3}-1)r+s]^2, \ (s\,;\mathrm{even}), \qquad (3.8\,\mathrm{a})$$

$$g^P_k(c,h,q)=2h-2kq+(\frac{c}{3}-1)(k^2-\frac{1}{4})-\frac{1}{4}, \ (k\in\boldsymbol{Z}). \qquad (3.8\,\mathrm{b})$$

For the T algebra, let $M^T_{\pm,n}$ be the matrix of inner products for the level n and fermion parity $(-1)^F=\pm 1$. On level 0 the determinant formulae are:

$$\det M^T_{+,0}=1, \quad \det M^T_{-,0}=h-\frac{c}{24}.$$

For $n>0$,

$$\det M^T_{\pm,n}(c,h)=(h-\frac{c}{24})^{\frac{P_T(n)}{2}} \prod_{1\le rs\le 2n,\, s;\,\mathrm{odd}} (f^T_{r,s})^{P_T(n-\frac{rs}{2})}, \qquad (3.9)$$

where

$$f^T_{r,s}(c,h)=2(\frac{c}{3}-1)(h-\frac{c}{24})+\frac{1}{4}[(\frac{c}{3}-1)r+s]^2, \ (s\,;\mathrm{odd}). \qquad (3.10)$$

The full unitary representations of the three classes of the $N=2$ superconformal algebra was conjectured by Boucher, Friedan and Kent [14] and proved in [14, 15]. (see Appendix A).

For the A algebra, the unitary repesentations fall into three classes:

A_3: $c\ge 3$, (c,h,q) such that $g^A_n\ge 0$ for all $n\in\mathbf{Z}+\frac{1}{2}$,

A_2: $c\ge 3$, (c,h,q) such that $g^A_n=0$, $g^A_{n+\mathrm{sgn}(n)}<0$, $f^A_{1,2}\ge 0$, for some $n\in\mathbf{Z}+\frac{1}{2}$,

A_0: $c<3$, $c=3(1-\frac{2}{m})$, $h_{jk}=\dfrac{(jk-\frac{1}{4})}{m}$, $Q^A_{jk}=\dfrac{(j-k)}{m}$,

for integer $m\ge 2$ and $j,k\in\boldsymbol{Z}+\frac{1}{2}$, $0<j,b,j+k\le m-1$.

For the P algebra, the unitary representations (with $\mathrm{sgn}(0)=\pm 1$) are

$P^{\pm}_3$: $c\ge 3$, (c,h,q) such that $g^P_n\ge 0$ for all $n\in\boldsymbol{Z}$,

$P^{\pm}_2$: $c\ge 3$, (c,h,q) such that $g^P_n=0$, $g^P_{n+\mathrm{sgn}(n)}<0$, $f^P_{1,2}\ge 0$ for some $n\in\boldsymbol{Z}$,

$P^{\pm}_0$: $c<3$, $c=3(1-\frac{2}{m})$, $h_{jk}=\frac{c}{24}+\frac{jk}{m}$, $Q^{P\pm}_{jk}=\pm(\frac{j-k}{m})$

for integer $m>2$ and $j,k\in\boldsymbol{Z}$, $0\le j-1,k,j+k\le m-1$.

For the T algebra, the unitary representations are

T_2: $c\ge 3$, $h\ge\frac{c}{24}$

T_0: $c<3$, $c=3(1-\frac{2}{m})$, $h_r=\frac{c}{24}+\frac{(m-2r)^2}{16m}$, for integers m, r such that $m\geq 2$ and $1\leq r\leq \frac{m}{2}$.

§ 4. The Character Formula for the $c<3$ Discrete Series

Now we will derive the character formula for the $c<3$ discrete series in terms of the unitary representations of the affine $SU(2)$ algebra with generators J^i_n, following the method of coset space construction of Goddard, Kent and Olive [3]. The coset structure of the $N=2$ discrete series is known as $SU(2)\times U(1)/U(1)$, which shows the apparent relationship between the $N=2$ discrete series and the unitary representations of affine Kac-Moody algebra $SU(2)$.

The $N=2$ supereconformal characters are explicitly calculated by Dobrev and Matsuo [16] by using the Kac determinant formula ($c<3$) to apply to write down the embedding relations of Verma modules of singular vectors derived by Rocha-Caridi [17].

For the A and $P^{\pm}$ sectors the character formulae are read as follows:

$$A: \chi^{A,(M)}_{jk}(\tau, z)=\varphi_A(\tau, z)q^{h^A_{jk}}y^{Q^A_{jk}}\Gamma^{(M)}_{jk}(\tau, z), \tag{4.1}$$

$$P^{\pm}: \chi^{P\pm,(M)}_{jk}(\tau, z)=\varphi_P(\tau,)q^{h^{P\pm}_{jk}}\, y^{Q^{P\pm}_{jk}}\,\Gamma^{(M)}_{jk}(\tau, z), \tag{4.2}$$

where

$$\varphi_A(\tau, z)=\prod_{n=1}^{\infty}\frac{(1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}})}{(1-q^n)^2}, \tag{4.3}$$

$$\varphi_P(\tau, z)=(y^{\frac{1}{2}}+y^{-\frac{1}{2}})\prod_{n=1}^{\infty}\frac{(1+yq^n)(1+y^{-1}q^n)}{(1-q^n)^2}, \tag{4.4}$$

and

$$\begin{aligned}\Gamma^{(M)}_{jk}(\tau, z)&=\sum_{n=0}^{\infty}q^{Mn^2+(j+k)n}\left[1-\frac{y^{-1}q^{Mn+j}}{1+y^{-1}q^{Mn+j}}-\frac{yq^{Mn+k}}{1+yq^{Mn+k}}\right]\\ &\quad-\sum_{n=1}^{\infty}q^{Mn^2-(j+k)n}\left[1-\frac{yq^{Mn-j}}{1+yq^{Mn-j}}-\frac{y^{-1}q^{Mn-k}}{1+y^{-1}q^{Mn-k}}\right]\\ &=\prod_{n=1}^{\infty}\frac{(1-q^{Mn+j+k-M})(1-q^{Mn-j-k})(1-q^{Mn})^2}{(1+yq^{Mn-j})(1+y^{-1}q^{Mn+j-M})(1+y^{-1}q^{Mn-k})(1+yq^{Mn+k-M})}.\end{aligned} \tag{4.5}$$

The last result is due to Matsuo.

For the T sector, the embedding pattern, is now the same as that of conformal algebra ($N=0$). The character formula is given:

$$\chi^{T,(M)}_r(\tau)=\varphi_T(\tau)q^{h^T_r}\,\tilde{\Gamma}^{(M)}_r(\tau), \tag{4.6}$$

where

$$\varphi_T(\tau)=\prod_{n=1}^{\infty}\frac{1+q^{\frac{n}{2}}}{1-q^{\frac{n}{2}}}, \tag{4.7a}$$

$$\widetilde{\Gamma}_r^{(M)}(\tau)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{Mn^2+n(M-2r)}{4}}$$

$$=\prod_{n=1}^{\infty}(1-q^{\frac{Mn}{2}})(1-q^{\frac{Mn-r}{2}})(1-q^{\frac{Mn-M+r}{2}}). \qquad (4.7\text{b})$$

Thus the characters of $N=2$ superconformal algebra are expressed in one infinite product forms, which fact distinguishes from those of conformal or $N=1$ superconformal algebra. However these characters seem too complicated to derive their modular group properties.

It is shown recently that the $N=2$ superconformal characters for the A and $P^{\pm}$ algebras can be written in terms of the string functions and the theta functions associated with affine $SU(2)$ Kac-Moody algebra [20]. It is convenient to work on using the quantum numbers (l, m) corresponding to the expressions obtained in the coset space construction [20] for the A and $P^{\pm}$ sectors:

$$A: l=j+k-1, \ m=j-k,$$

$$P^{\pm}: l=j+k-1, \ m=\pm(j-k-1),$$

where $l-m=0$ (mod 2) and $0\leq l\leq p$ with $M=p+2$. Correspondingly the replacement is implied as:

$$h_{jk}^{(\cdot)}\longleftrightarrow h_{lm}^{(\lambda)},$$

$$Q_{jk}^{(\cdot)}\longleftrightarrow Q_{lm}^{(\lambda)},$$

and

$$\chi_{jk}^{(M,\cdot)}\longleftrightarrow \chi_{lm}^{(p,\lambda)},$$

where $\lambda=0$, (or $\mp\frac{1}{2}$) for the A (or $P^{\pm}$) aglebra. Now the central charge, the conformal weight and $U(1)$ charge are given as:

$$c=3(1-\frac{2}{p+2}),$$

$$h_{lm}^{(\lambda)}=\frac{l(l+2)}{4(p+2)}+\frac{\lambda^2}{2}-\frac{(m-2\lambda)^2}{4(p+2)}, \qquad (4.8)$$

$$Q_{lm}^{(\lambda)}=\frac{m+\lambda p}{p+2},$$

where $\frac{l}{2}$ and $\frac{m}{2}$ are the isospin and the third isospin component in the level p $SU(2)$ Kac-Moody algebra respectively and λ is a $U(2)$ charge [19, 20, 16].

The character formula shown in eqs. (4.1) and (4.2) are rewritten in terms of the string functions and the theta functions as:

$$\chi_{l,m}^{(p,\lambda)}(\tau,z)=\sum_{m'=-p+1}^{p}c_{l,m'}^{(p)}(\tau)\Theta_{m'(p+2)-mp+2\lambda p,\,p(p+2)}\left[\frac{\tau}{2},\frac{z}{p+2}\right], \qquad (4.9)$$

where the classical theta functions $\Theta_{m,k}$ are defined by

$$\Theta_{m,k}(\tau,z)=\sum_{n\in\mathbf{Z}+\frac{m}{2k}}\exp[2\pi ik(\tau n^2-nz)]$$

$$= \sum_{n\in\mathbf{Z}+\frac{m}{2k}} q^{kn^2}y^{-kn}. \tag{4.10 a}$$

and $c^{(p)}_{l,m}(\tau)$ are the string functions of the affine $SU(2)$ algebra which are Hecke modular forms [19, 39],

$$c^{(p)}_{l,m}(\tau)=\eta(\tau)^{-3} \sum_{\substack{-|x|<y\leq|x| \\ (x,y) \text{ or } (\frac{1}{2}-x,\frac{1}{2}+y)\in\left(\frac{l+1}{(2p+2)},\ \frac{m}{2p}\right)+\mathbf{Z}^2}} \operatorname{sgn}(x)q^{[(p+2)x^2-py^2]}, \tag{4.10 b}$$

where $\eta(\tau)$ is the Dedekind's η-function defined later.

First we derive the formula in the case of the A algebra. Applying Jacobi triple product identity

$$\sum_{n\in\mathbf{Z}} y^n q^{n^2}=\prod_{n=1}^{\infty}(1-q^{2n})(1+yq^{2n-1})(1+y^{-1}q^{2n-1}), \tag{4.11}$$

to (4.1) we obtain,

$$\begin{aligned}\chi^{(P,0)}_{jk}(\tau,z)&=\prod_{n=1}^{\infty}\frac{(1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}})}{(1-q^n)^2}q^{h^{(0)}_{lm}-\frac{c}{24}}y^{Q^{(0)}_{lm}}\Gamma^{(M,A)}_{lm}(\tau,z)\\ &=q^{-\frac{p(p+1)}{24}+\frac{(p-2l)^2}{8(p+2)}}\prod_{n=1}^{\infty}\frac{(1-q^{Mn+j+k-M})(1-q^{Mn-j-k})}{(1-q^{Mn})^{M-4}}\\ &\quad\times\prod_{r=0}^{M-1\,\prime}\Theta_{M-1-2r,M}\left(\frac{\tau}{2},\frac{z}{M}\right),\end{aligned} \tag{4.12}$$

where $\Pi^{M-1\prime}_{r=0}$ is understood to exclude $r=\dfrac{2j-1}{2}$ and $r=M-\dfrac{2k+1}{2}$. Using the multiplication formula of theta functions for $M\geq 4$ [39]

$$\Theta_{m,k}(\tau,z)\Theta_{m'k'}(\tau,z)=\sum_{l=1}^{k+k'}\Theta_{mk'-m'k+2lkk',kk'(k+k')}(\tau,0)\Theta_{m+m'+2lk,k+k'}(\tau,z), \tag{4.13}$$

we obtain

$$\chi^{(p,0)}_{l,m}(\tau,z)=\sum_{m'=-p+1}^{p}\tilde{c}^{(p)}_{lm'}(\tau)\Theta_{m'(p+2)-mp,p(p+2)}\left[\frac{\tau}{2},\frac{z}{p+2}\right], \tag{4.14}$$

where we have switched (j,k) to (l,m) and

$$\begin{aligned}\tilde{c}^{(p)}_{l,m}(\tau)q^{-\frac{p(p+1)}{24}+\frac{(p-2l)^2}{8(p+2)}}&\prod_{n=1}^{\infty}\frac{(1-q^{(p+2)n+l-p-1)})(1-q^{(p+2)n-l-1})}{(1-q^n)^2(1-q^{(p+2)n})^{p-2}}\\ &\times\sum_{\{r_b\}}\prod_{b=1}^{p-1}\Theta_{\mu(b,p),b(b+1)(p+2)}(\tau,0),\end{aligned} \tag{4.15}$$

$$\{r_a\}=\left\{r_a\,|\,r_a=0,1,\cdots,a,\ \ a=1,\cdots,p-1 \text{ with } \sum_{a=1}^{p-1}ar_a=\frac{m'-m}{2},\ \bmod p\right\},$$

$$\mu(b,p)=\sum_{a=1}^{b}m_a-bm_{b+1}+2(p+2)\sum_{a=1}^{b}ar_a,$$

$$\{m_a\}=\left\{p+1-2r\,|\,r=0,\cdots,p+1 \text{ excluding } \frac{l+m}{2} \text{ and } p+1-\frac{l-m}{2}\right\}.$$

For $M=3$, $(p=1)$

$$\tilde{c}^{(1)}_{l,m}(\tau)=\begin{cases}\eta(\tau)^{-1}, & \text{for } l-m \bmod 2;\\ 0, & \text{otherwise,}\end{cases} \tag{4.16}$$

where $\eta(\tau)$is the Dedekind's η-function

$$\eta(\tau)=q^{\frac{1}{24}}\prod_{n=1}(1-q^n),$$

which is a modular form of weight 1/2.

Comparing with the explicit formulae of Kac-Peterson [39] for $c^{(p)}_{lm}$, one can check that

$$\tilde{c}^{(p)}_{lm}=c^{(p)}_{lm}, \tag{4.17}$$

for lower value of p. The general validity of this formula (4.9) are conjectured in ref. [19] by means of different context. The characters satisfy the relations that are useful later:

$$\chi^{(p,\lambda)}_{lm}=\chi^{(p,\lambda)}_{p-l,\,p+2+m}=\chi^{(p,\lambda)}_{p,\,m+2(p+2)\mathbf{Z}},$$
$$\chi^{(p,\lambda)}_{l,-m}(\tau,z)=\chi^{(p,-\lambda)}_{l,m}(\tau,-z), \tag{4.18}$$

and

$$\chi^{(p,\lambda)}_{lm}=0,$$

whenever $l-m\neq 0 \bmod 2$.

Constructing the unitary highest weight representations of $N=2$ algebra by combining the parafermionic Z_p symmetric conformal field theory [18, 19] and a free massless boson system without interactions, the character formulae are obtained more physically [20].

§ 5. The Transformation Laws of the Characters of the A and P Algebras under the Modular Group

In this section we will investigate the modular transformations of the characters of A and P algebras. For the twisted algebra T we will show only the character coincidence between T and $SU(2)$ Kac-Moody algebra in § 7.

The modular transformations on the torus are given by

$$\tau\to\frac{a\tau+b}{c\tau+d},\ ad-bc=1 \text{ and } a,b,c,d\in\mathbf{Z}, \tag{5.1a}$$

which is generated by

$$T:\tau\to\tau+1,$$
$$S:\tau\to-\frac{1}{\tau}. \tag{5.1b}$$

The transformation laws of the string functions are determined from those of the affine $SU(2)$ characters [19]

$$\chi_{l,k}(\tau, z) = \sum_{m=-k+1}^{k} c_{lm}^{(k)}(\tau)\Theta_{m,k}(\tau, z)$$
$$= \frac{\Theta_{l+1,k+2} - \Theta_{-l-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}}, \tag{5.2}$$

and those of the symmetry properties as

$$c_{lm}^{(k)} = c_{l,m+2k\mathbf{Z}}^{(k)} = c_{l,-m}^{(k)} = c_{k-l,k-m}^{(k)},$$
$$c_{lm}^{(k)} = 0,$$

if $l-m \neq 0$, mod 2,

$$\theta_{m,k} = \theta_{m+2k\mathbf{Z},k}. \tag{5.3}$$

They are derived as follows [20]

$$T: c_{lm}^{(p)}(\tau+1) = e^{2\pi i\left[\frac{l(l+2)}{4(p+2)} - \frac{m^2}{4p} - \frac{c}{24}\right]} c_{lm}^{(p)}(\tau),$$
$$S: c_{lm}^{(p)}\left(-\frac{1}{\tau}\right) = (-i\tau)^{-\frac{1}{2}} \sum_{l'=0}^{p} \sum_{m'=-p+1}^{p} S_{l,l'} T_{m,m'}^{(p)} c_{l',m'}^{(p)}(\tau), \tag{5.4}$$

where

$$T_{m,m'}^{(p)} = \frac{1}{\sqrt{2p}} e^{i\pi\frac{mm'}{p}},$$
$$S_{l,l'} = \sqrt{\frac{2}{p+2}} \sin\frac{\pi(l+1)(l'+1)}{p+2}. \tag{5.5}$$

Then by combining the above transformation formulae with those of theta functions,

$$T: \Theta_{m,k}(\tau+1, z) = e^{i\pi\frac{m^2}{2k}}\Theta_{m,k}(\tau, z),$$
$$S: \Theta_{m,k}\left(-\frac{1}{\tau}, \frac{z}{\tau}, u + \frac{z^2}{2k^2\tau}\right) = \frac{1}{\sqrt{2k}}(-i\tau)^{\frac{1}{2}} \sum_{l \bmod 2k} e^{-i\pi\frac{lm}{k}}\Theta_{lm}(\tau, z, u), \tag{5.6}$$

we can obtain the modular transformation formula for the characters under T and S. If we define the twisted (or projected) characters by,

$$\chi = Tr[(-1)^F q^{L_0 - \frac{c}{24}}], \tag{5.7}$$

then the explicit formula is given as:

$$\chi_{lm}^{(p,\tilde{0})}(\tau, z) = e^{2i\pi\left[h_{lm}^{(0)} \mp \frac{m}{2(p+2)} - \frac{c}{24}\right]} \sum_{m'=-p+1}^{p} c_{lm'}^{(p)} \Theta_{m'(p+2)-mp,p(p+2)}\left[\frac{\tau}{2}, \frac{z \pm \frac{1}{2}}{p+2}\right], \tag{5.8}$$

where $\tilde{0}$ denotes the twisted state of the A algebra. Now we can show the modular transformations of characters under T,

$$\begin{aligned} \chi_{lm}^{(p,0)}(\tau+1, z) &= \chi_{lm}^{(p,\tilde{0})}(\tau, z), \\ \chi_{lm}^{(p,\tilde{0})}(\tau+1, z) &= e^{2i\pi\left[h_{lm}^{(0)}-\frac{c}{24}\right]} \chi_{lm}^{(p,0)}(\tau, z), \\ \chi_{lm}^{(p,\pm\frac{1}{2})}(\tau+1, z) &= e^{2i\pi\left[h_{lm}^{(\pm\frac{1}{2})}-\frac{c}{24}\right]} \chi_{lm}^{(p,\pm\frac{1}{2})}(\tau, z). \end{aligned} \tag{5.9}$$

While under S,

$$\begin{aligned} \chi_{lm}^{(p,0)}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= e^{i\pi\frac{pz^2}{p+2}} \sum_{l'=0}^{p} \sum_{m'=-p-1}^{p+2} S_{ll'}\, T_{-m,m'}^{(p+2)}\, \chi_{l',m'}^{(p,0)}(\tau, z), \\ \chi_{lm}^{(p,\tilde{0})}\left(-\frac{1}{\tau}, 0\right) &= e^{2i\pi\left[h_{lm}^{(0)}-\frac{c}{24}\right]} \sum_{l'=0}^{p} \sum_{m'=-p-1}^{p+2} S_{ll'}\, T_{-m,m'\mp 1}^{(p+2)}\, \chi_{l'm'}^{(p,\pm\frac{1}{2})}(\tau, 0), \\ \chi_{lm}^{(p,\pm\frac{1}{2})}\left(-\frac{1}{\tau}, 0\right) &= \sum_{l'=0}^{p} \sum_{m'=-p-1}^{p+2} S_{ll'}\, T_{-(m\mp 1),m'}^{(p+2)}\, e^{-2i\pi\left[h_{l'm'}^{(0)}-\frac{c}{24}\right]} \chi_{l'm'}^{(p,\tilde{0})}(\tau, 0). \end{aligned} \tag{5.10}$$

Setting $z=0$, it is clear that a set of characters $\chi_{lm}^{(p,\lambda)}$ with $\lambda=0, \tilde{0}, \frac{1}{2}\left(\text{or} -\frac{1}{2}\right)$ carries the unitary representation of the modular group and can be understood as different spin structures transforming into each other [34].

§ 6. The A-D-E Classfications of the $N=2$ Superconformal Theory

Modular invariant partition functions of $N=2$ superconformal field theories on a torus are obtained by summing over different structures with the Neveu-Schwarz (NS) and Ramond(R) algebras, which are specified by antiperiodic and periodic boundary conditions on the supercurrent on a cylinder.

The partition function can be given by,

$$\begin{aligned} Z &= \frac{1}{2}\left|Tr q^{L_0-\frac{c}{24}}\right|^2 + \frac{1}{2}\left|Tr(-1)^F q^{L_0-\frac{c}{24}}\right|^2 \\ &= \frac{1}{2}[Z_{NS}(\tau)+Z_{\widetilde{NS}}(\tau)] + \frac{1}{2}[Z_R(\tau)+\eta Z_{\tilde{R}}(\tau)], \end{aligned} \tag{6.1}$$

where $Z_{\widetilde{NS}}$ and Z_R are the NS and R sectors respectively, twisted by the fermion parity operator $(-1)^{F+\bar{F}}$. In the R sector one can choose either fermion number even ($\eta=+1$) or odd ($\eta=-1$) projection. Since the R states are twofold degenerate with respect to $(-1)^F$ except the lowest state $\left(\frac{c}{24}, \frac{c}{24}\right)$, the cancellation between degenerate states yields

$$Z_{\tilde{R}} = Tr_{[R]}(-1)^{F+\bar{F}}. \tag{6.2}$$

Using the infinite product representation of the character $\chi_{lm}^{(p,\lambda)}(\tau, z)$, an identity is found in the R sector as:

$$\chi_{lm}^{(p,-\frac{1}{2})} = \delta_{lm}, \tag{6.3}$$

where the conformal weight is given

$$h_{ll}^{(p,-\frac{1}{2})}=\frac{c}{24}, \tag{6.4}$$

for $l=0,1,\cdots,p$. This is the consequnce of the supersymmetric pairing in the R-excited states. The unpaired lowest energy states with $h=\frac{c}{24}$ survive the cancellation, yielding a constant. This is the Witten index in the R sector ($P_0^{\pm}$ sector)[32]. The partition function is now written in terms of the characters [20],

$$Z(\tau)=\sum_{l,\bar{l}=0}^{p}\sum_{m,\bar{m}=-p-1}^{p+2}\sum_{\lambda=0,\tilde{0},\frac{1}{2}}[N_{l,m;\bar{l},\bar{m}}^{(p,\lambda)}\chi_{lm}^{(p,\lambda)}(\tau)\chi_{\bar{l}\bar{m}}^{(p,\lambda)}(\tau^*)+\eta\chi_{lm}^{(p,-\frac{1}{2})}\chi_{\bar{l}\bar{m}}^{(p,-\frac{1}{2})*}], \tag{6.5}$$

where $N_{l,m;\bar{l},\bar{m}}^{(p,\lambda)}\in\mathbf{Z}_+$.

Now we are ready to classify the modular invariant partition functions. Invariance under T lead to the conditions:

$$h_{l,m}^{\pm\frac{1}{2}}-h_{\bar{l},\bar{m}}^{\pm\frac{1}{2}}\in\mathbf{Z},\ h_{l,m}^{(0)}-h_{\bar{l},\bar{m}}^{(0)}\in\frac{\mathbf{Z}}{2},$$

and

$$N^{(0)}=N^{(\tilde{0})}. \tag{6.6}$$

The invariance under S implies:

$$N_{l',m';\bar{l}',\bar{m}'}^{(0)}=S_{l,l'}S_{\bar{l},\bar{l}'}T_{-m,m'}^{(p+2)}T_{\bar{m},\bar{m}'}^{(p+2)}N_{l,m;\bar{l},\bar{m}}^{(0)}, \tag{6.7 a}$$

$$N_{l',m';\bar{l}',\bar{m}'}^{(\frac{1}{2})}=S_{l,l'}S_{\bar{l},\bar{l}'}T_{-m,m'-1}^{(p+2)}T_{\bar{m},\bar{m}'-1}^{(p+2)}e^{2\pi i\left(h_{l,m}^{(0)}-h_{\bar{l},\bar{m}}^{(0)}\right)}N_{l,m;\bar{l},\bar{m}}^{(0)}, \tag{6.7 b}$$

where the summation over $l,\bar{l},m,\bar{m}$ are implicitly understood. The solutions of these equations provide the classification of the $N=2$ superconformal field theories. There is a factorization of the superconformal characters into three natural pieces, as observed by Gepner. The transformation of the character $\chi_{l;m}^{(q;\lambda)}$ contains l piece which is just an affine $A_1^{(1)}$ transformation, a m piece which is a level $p+2$ theta function system, and λ piece which is a level 2 theta function. Then a modular invariant solution for the $N=2$ system will be given by,

$$N_{l,m;\bar{l},\bar{m}}^{(\lambda)}=N_{l,\bar{l}}Q_{m,\bar{m}}S_{\lambda,\lambda'}. \tag{6.8}$$

Then all the modular invariant solutions can be written as:

$$N_{l,m;\bar{l},\bar{m}}^{(0)}=N_{l,\bar{l}}Q_{m,\bar{m}},$$

$$N_{l,m;\bar{l},\bar{m}}^{(\frac{1}{2})}=N_{l,\bar{l}}Q_{m-1,\bar{m}-1}. \tag{6.9}$$

The $N_{l,\bar{l}}$ describes modular invariant in the level p $SU(2)$ WZW models [5, 6, 7],

$$Z_{WZW}^{G}(\tau)=\sum_{l,\bar{l}=0}^{p}N_{l\bar{l}}^{G}\chi_l(\tau)\chi_{\bar{l}}^*(\tau), \tag{6.10}$$

which are classified by simply laced Lie algebras G with Coxester number $p+2$ [8], and χ_l's are $SU(2)$ characters given as:

$$\chi_l(\tau)=\sum_{m=-p+1}^{p} c_m^l(\tau)\Theta_{m,p}(\tau). \tag{6.11}$$

The symmetry properties are given:

$$c_m^l=c_{-m}^l=c_{m+2p}^l=c_{p-m}^{p-l},$$
$$N_{l,\bar{l}}^G=N_{p-l,p-\bar{l}}^G. \tag{6.12}$$

The complete system of invariants for the theta system is shown in ref. [19]. The partition function is given by,

$$Z_{p+2}=|\eta|^{-2}\sum_{m,\bar{m}} Q_{m,\bar{m}}\Theta_{m,p+2}\Theta_{\bar{m},p+2}^*. \tag{6.13}$$

Suppose that α and β are two integers such that $\alpha\beta=p+2$. Then every such pair (α,β) corresponds to a unique solution at level $p+2$, denoted by $Q^{\alpha,\beta}$. We have then,

$$Q_{m,\bar{m}}^{\alpha,\beta}=\frac{1}{2}\sum_{x\in\mathbf{Z}_{2\beta};\,y\in\mathbf{Z}_{2\alpha}}\delta_{m,\alpha x+\beta y}\,\delta_{\bar{m},\alpha x-\beta y}. \tag{6.14}$$

For example the diagonal solution (Ising model) is $(1,p+2)$ solution, which exists for any level. Now we complete the classification of the modular invariant partition functions for a minimal theory of the A_0 and $P_0^{\pm}$ algebra among the $N=2$ superconformal field theories, since any such invariants is composed out of an $A_1^{(1)}$ invariant [8] and two theta function invariants [19].

Finally the Witten index is evaluated by S. K. Yang [22],

$$Tr_{[R]}(-1)^{F+\bar{F}}=\text{rank of } G, \tag{6.15}$$

indicating the (2, 2) supersymmetry unbroken in the unitary minimal series [14, 31].

§ 7. A Connection between the Twisted $N=2$ Superconformal and $SU(2)$ Kac-Moody Algebras

Recently Rittenberg and Schwimmer have shown that there is a one-to-one correspondence between the unitary highest weight representations of the (twisted) $SU(2)$ Kac-Moody algebra and the discrete series of unitary representations of the twisted $N=2$ superconformal algebra.

The twisted Kac-Moody algebra is defined:

$$[J_n^\alpha, J_m^\beta]=i\varepsilon_{\alpha\beta\gamma}J_{n+m}^\gamma+k\delta^{\alpha\beta}n\delta_{n+m,0}, \tag{7.1}$$

where for J_n^1, J_n^2, $n\in\mathbf{Z}+\frac{1}{2}$ and for J_n^3, $n\in\mathbf{Z}$. This algebra is isomorphic to the usual (untwisted) $SU(2)$ algebra, generated by S_n^α with $n\in\mathbf{Z}$ for $\alpha=1,2,3$:

$$S_n^{\pm}=J_{n\pm\frac{1}{2}}^{\pm},\quad S_n^3=J_n^3+\frac{1}{2}k\delta_{n,0}. \tag{7.2}$$

A Virasoro algebra is constructed by the Sugawara form:

$$L_n^J \equiv \frac{1}{2k+2} \sum_{\alpha,m} : J_{n+m}^\alpha J_{-m}^\alpha :, \tag{7.3}$$

where the indices take the values specified above and the normal ordering: puts operators with positive indices to the right. Through the isomorphism (7.2) L_n^J can be expressed as

$$L_n^J = L_n^S - \frac{1}{2} S_n^3 + \frac{1}{8} k, \tag{7.4}$$

where

$$L_n^S = \frac{1}{2k+2} \sum_{\alpha,m} : S_{n+m}^\alpha S_{-m}^\alpha :. \tag{7.5}$$

A unitary representation of the S-algebra is chracterized by a value k and a highest weight $I=0, \frac{1}{2}, \cdots, k$. Through (7.4) and (7.5) this produces a reducible representation of the L^J Virasoro algebra with central charge

$$c = \frac{3k}{k+2} = 3\left(1 - \frac{2}{m}\right), \tag{7.6}$$

and the highest weight

$$h_{k,I} = \frac{(m-2r)^2}{16m} + \frac{c}{24}, \tag{7.7}$$

where

$$m \equiv 2k+2, \; r \equiv 2I+1, \; 1 \leq r \leq m-1. \tag{7.8}$$

Both L_n^J and L_n^S are physically different, since L_0^S commutes with all the generators S_n^α while L_0^J commutes only with S_n^3. The twisted $N=2$ superconformal algebra has been shown at §2 and §3. Comparing (3.9) with (7.6), (7.7), we notice that there is a one-to-one correspondence between the representations T_0 and the unitary highest weight representations of the affine $SU(2)$ algebra.

Now we will show the correspondence more precisely by calculating the characters of the representations. For the $SU(2)$ Kac-Moody algebra, the character is given by:

$$\chi_{k,I}(z) \equiv Tr[z^{L_0^J}], \tag{7.9}$$

where L_0^J is given by (7.4). Using the Weyl-Kac formula the explicit form of the character has been given in ref. [5]:

$$\chi(z) = z^{h_{k,I}} M(z) \varphi_{k,I}(z), \tag{7.10}$$

where

$$M(z) \equiv \prod_{n=1}^{\infty} (1-z^n)^{-1} (1-z^{n-\frac{1}{2}})^{-2}, \tag{7.11}$$

$$\varphi_{k,I}(z) \equiv \sum_n \left[z^{mn^2 + (r-\frac{m}{2})n} - z^{mn^2 + n(r+\frac{m}{2}) + \frac{r}{2}} \right], \tag{7.12}$$

and $h_{k,l}$, m and r are given by (7.7), (7.8). The character formula for the T_0 series has been shown (4.6), (4.7a, b) at §4. It is easy to show that $\varphi_T=M_{k,l}$, $\Gamma_r^{(M)}=\varphi_{k,l}$ and therefore the character coincidence between the twisted $N=2$ superconformal and $SU(2)$ Kac-Moody algebras is proved [22, 23], i.e.,

$$\chi_{k,l}(z)=\chi_{m,r}(z). \tag{7.13}$$

Thus the corresponding representations can be realized on the same Hilbert space originated from the same Virasoro algebra. As a consequence the generators of one algebra can be expressed in terms of the generators of the other algebra, whose connections though depend on the representations. Thus it is not clear if the unitary representations of the $N=2$ superconformal algebra belong to the class T_0 can be mapped into some class of $SU(2)$ representations.

§ 8. String Compactification

The construction of the heterotic string compactifications out of tensor product of $c<3$ minimal models seems one of the most interesting applications of the $N=2$ superconformal field theory [20, 32]. The importance of preserving $N=1$ space-time supersymmetry in a classical superstring ground state has been emphasized by many authors, because it seems to be the only probability of solving the hierarchy problem in a theory. It has been shown that a necessary and sufficient criterion is the existence of an $N=2$ superconformal symmetry on the two-dimensional world sheet, with a charge quantization condition on the $U(1)$ current in this algebra [31].

The $N=2$ superconformal algebra admit a generalization in which G^+ and G^- have twisted boundary conditions by opposite phases, where $G^\pm$ are defined as $G^\pm=\frac{1}{\sqrt{2}}(G^1\pm iG^2)$ in eq. (2.2b), i.e.,

$$\begin{aligned} G^i(e^{2\pi i}z)&=U^{ij}G^j,\\ J(e^{2\pi i}z)&=(\det U)J(z), \end{aligned} \tag{8.1}$$

where

$$U^{ij}\in O(2).$$

There is also an automorphism $SO(2)$:

$$\begin{aligned} T(z)&\to T(z)-iz\partial_z\alpha(z)J(z)-\frac{c}{6}[z\partial_z\alpha(z)]^2,\\ J(z)&\to J(z)-i\frac{c}{3}z\partial_z\alpha(z),\\ G^\pm(z)&\to e^{\pm i\alpha(z)}G^\pm(z), \end{aligned} \tag{8.2}$$

where α is defined by,

$$G^i(z)\to[e^{i\alpha(z)J}]_{ij}G^j(z),$$

with $SO(2)$ generator J. Thus there are two essentially distinct sectors in the $N=2$ superconformal algebra, because of $O(2)/SO(2)=\mathbf{Z}_2$. For example, let us define

$$G^{\pm}(e^{2\pi i}z)=e^{\pm 2\pi i\eta}G^{\pm}(z), \tag{8.3}$$

then we can obtain the η-algebra generated by,

$$\begin{aligned} L_n^{(\eta)}&=L_n^{(0)}+\eta J_n^{(0)}+\frac{c}{6}\eta^2\delta_{n,0},\\ J_n^{(\eta)}&=J_n^{(0)}+\frac{c}{3}\eta\delta_{n,0},\\ G_{r\mp\eta}^{\pm(\eta)}&=G_r^{\pm(0)}, \end{aligned} \tag{8.4}$$

where $n\in\mathbf{Z}$ and $r\in\mathbf{Z}+\frac{1}{2}$. The η-algebras satisfy the eq. (2.3), and are all equivalent. We denote this algebra as A_η. There is an isomorphism between $A_{\eta+1}$ and A_η. There is an infinite number of $N=2$ superconformal algebras corresponding to difierent choices of η. In particular the algebra is in the Neveu-Schwarz sector if $\eta=0$, and in the Ramond sector if $\eta=\frac{1}{2}$. Either $\eta=0$ or $\eta=1$ describe the same sector of the theory, but a field at $\eta=0$ is mapped to another field at $\eta=1$ in terms of the original $\eta=0$ algebra. This is called spectral flow [36].

Now we concentrate on constructing the exactly solvable models of the superstring compactifications following to Gepner. Suppose that the dimension of Minkowski space is $D=d+2$, where d is the number of transverse dimensions, which we restrict d to be even. The total trace anomaly will be $c=12$ in the light cone gauge, in both the left and right sectors of the theory. Hence the internal space will have a trace anomaly $c_0=12-\frac{3}{2}d$, which can be made as a sum of the minimal models,

$$c_0=\frac{3}{2}(10-D)=\sum_{i=1}^{l}\frac{3k_i}{k_i+2}. \tag{8.5}$$

The full theory in the light cone gauge has an $N=2$ supereconformal structure in this case. The fact that the theory has $N=2$ world sheet SUSY opens the way for a simple mechanism for achieving spece-time supersymmetry. The proof that the $N=1$ space-time SUSY implies the $N=2$ world-sheet SUSY is reinterpreted in terms of the spectral flow. Now consider the flow of the identity field I, $(h=Q=0)$ at $\eta=0$ to $\eta=\pm\frac{1}{2}$. The supercharges $Q_\alpha, Q_{\dot\alpha}$ of the space-time SUSY is expressed by the gravitino vertex $V_\alpha^{(-\frac{1}{2})}$ and $V_{\dot\alpha}^{(-\frac{1}{2})}$ respectively, i.e.,

$$\begin{aligned} Q_\alpha&=\oint dz V_\alpha^{(-\frac{1}{2})}(z),\\ Q_{\dot\alpha}&=\oint dz V_{\dot\alpha}^{(-\frac{1}{2})}(z). \end{aligned} \tag{8.6}$$

The gravitino vertices are given by,

$$V_{\alpha}^{(-\frac{1}{2})}(z)=e^{-\frac{\phi}{2}}S_{\alpha}\Sigma, \tag{8.7a}$$

and

$$V_{\dot{\alpha}}^{(-\frac{1}{2})}(z)=e^{-\frac{\phi}{2}}S_{\dot{\alpha}}\Sigma^{\dagger}, \tag{8.7b}$$

where $e^{-\frac{\phi}{2}}$ is the spin field for (β,γ) ghost system, S_{α} is the spin field for the world sheet fermions ψ^{μ} and Σ is some unknown field in the Ramond sector with $h=\frac{3}{8}$, which correspond to the covarintly constant spinor field on the Calabi–Yau manifold. The Σ and $\Sigma^{\dagger}$ can be obtained from the identity field of the NS sector by the spectral flow in cases of $\eta=\pm\frac{1}{2}$ and $c=9$ by the relations,

$$h'=h\pm\frac{Q}{2}+\frac{3}{8},\ Q'=Q\pm\frac{3}{2}. \tag{8.8}$$

Then eqs. (8.6) define the generators of space-time supersymmetry, if the $U(1)$ charges of all the fields in the theory are integers. In that case the gravitino vertex operator is local with respect to all the fields in the theory, and the action of the space-time supercharge on the fields will be equivalent to a flow from $\eta=0$ to $\eta=\pm\frac{1}{2}$, thus pairing fermionic and bosonic states in the actual spectrum of the theory. This mechanism guarantee the space-time supersymmetry. From Σ and $\Sigma^{\dagger}$ we can get $O(h=\frac{3}{2},\ Q=3)$ and $O^{\dagger}(h=\frac{3}{2},\ Q=-3)$ respectively which express the original NS sector, but with different highest weight states that correspond to the holomorphic or anti-holomolphic 3-form on the Calabi–Yau manifold. The generators of the flow $\Delta\eta=\pm 1$ are conformal fields with the same h and Q. In the case of $D=6$, they are nothing but the $SU(2)$ currents $J^{\pm}$. When the $N=2$ algebra is extended by the addition of the flow generators $J^{\pm}$, the $N=4$ algebra is obtained. Thus the string compactifications on the K_3 surface are described by the representation theory of the $N=4$ algebra. On the other hand, when $D=4$, in the case of Calabi–Yau compactification, the flow generators are fermionic and a new algebra is obtained by their addition to the $N=2$ algebra introduced by Zamolodchikov [26].

Next we will find the way to make the charge integrality condition compatible with modular invariance by means of β-method. The primary field $\phi_{(1,q,s)}$ of the $c=\sum_{i=1}^{r}c_i=9$ theory satisfies the relation:

$$\phi_{(1,q,s)}=\phi_{(l_1,q_1,s_1)}\cdots\phi_{(l_r,q_r,s_r)}, \tag{8.9}$$

where $\phi_{(l_i,q_i,s_i)},(i=1,\cdots,r)$ are the primary fields of the minimal theories, in which c_i, h_i and Q_i are defined by

$$\begin{aligned} c_i&=\frac{3k_i}{k_i+2},\\ h_i&=\frac{l_i(l_i+2)-q_i^2}{4(k_i+2)}+\frac{s_i^2}{8},\\ Q_i&=-\frac{q_i}{k_i+2}+\frac{s_i}{2}, \end{aligned} \tag{8.10}$$

where $0 \leq l_i \leq k_i$, $q_i \in \mathbf{Z}/2(k_i+2)\mathbf{Z}$, $s_i \in \mathbf{Z}/4\mathbf{Z}$, $l_i+q_i+s_i=$even and

$$h=\sum_{i=1}^{r} h_i, \quad Q=\sum_{i=1}^{r} Q_i.$$

Now we define the orbits of the spectral flow of the primary field by means of a vector β:

$$\phi_{(\mathrm{l},\mathrm{q},\mathrm{s})} \to \phi_{(\mathrm{l},\mathrm{q},\mathrm{s})+\beta} \to \cdots \to \phi_{(\mathrm{l},\mathrm{q},\mathrm{s})+n\beta} \to \cdots, \tag{8.11}$$

where β is expressed by

$$\beta=(0, \cdots, 0\,; 1, \cdots\,; 1, \cdots).$$

It is necessary to include the whole fields obtained in each orbit in order to guarantee the compatibility of space-time SUSY and modular invariance in the theory. Corresponding to a primary field, we can define a character by:

$$\chi_{(\mathrm{l},\mathrm{q},\mathrm{s})}=\chi_{(l_1,q_1,s_1)} \cdots \chi_{(l_r,q_r,s_r)}. \tag{8.12}$$

We will introduce the character of the primary fields belonging to the same orbit by:

$$\chi_I=\sum_n \chi_{(\mathrm{l},\mathrm{q},\mathrm{s})+n\beta}, \tag{8.13}$$

where I label the whole distinct orbits. Then we can define the partition function of a nonlinear σ model on Calabi-Yau manifold which is modular invariant:

$$Z(\tau)=\sum_{I,\bar{I}} N_{I,\bar{I}} \chi_I \chi_{\bar{I}}^{*}. \tag{8.14}$$

The contributions of space-time oscillators X^μ and ψ^μ, $(\mu=1,2)$ are included into the whole theory by employing the level 2 theta functions. Since the $SO(2)$ is a little group of the $SO(3,1)$ Lorentz group, the four helicity states are classified to a singlet$(O, h=0)$, a vector$(V, h=\frac{1}{2})$, a spinor $(s, h=\frac{1}{8})$ and a conjugate-spinor $(\bar{s}, h=\frac{1}{8})$. The characters of these four states are expressed by those of level 1 $SO(2)$ Kac-Moody algebra,

$$\chi_a^{SO(2)}=\eta^{-1}\theta_{s,2}, \tag{8.15}$$

where $\Theta_{s,2}$, $(s=0,1,2,3)$ are level 2 theta functions with $a=O, s, V, \bar{s}$ respectively. Then the total contribution to the character of the SST from the space-time oscillators and the compactified space will be given by,

$$\chi_I^{SST}=\chi_V^{SO(2)} \chi_I^{(s=0)}+\chi_O^{SO(2)} \chi_I^{(s=2)}-\chi_s^{SO(2)} \chi_I^{(s=1)}-\chi_{\bar{s}}^{SO(2)} \chi_I^{(s=3)}. \tag{8.16}$$

Now the modular invariant partition function for the superstring compactification is described by,

$$Z(\tau)=\sum_{I,\bar{I}} N_{I,\bar{I}} \chi_I^{SST} \chi_{\bar{I}}^{SST\,*}. \tag{8.17}$$

Thus we have found a superstring partition function for a compactification on an arbitrary sum of minimal conformal models, with $N=1$ space-time supersymmetry. In addition the theories have the correct connection between spin and statistics. The absence of tachyons is also guaranteed by the fact that we have space-time supersymmetry.

For the more physically interesting compactification of the heterotic string, we only mention on the results and leave the details to the original papers by Gepner [20, 32] ane others [33, 37, 38] for the further developments and applications. States in the heterotic compactification represented by $V_R \otimes V_L$ where $V_R=(w\,;q_1,\cdots,q_r\,;s_1,\cdots,s_r)$ and $V_L=(\bar{w}\,;\bar{q}_1,\cdots,\bar{q}_r\,;\bar{s}_1,\cdots,\bar{s}_r)$, in which $w(\bar{w})$ stands for an $SO(2)(SO(10))$ weight. The mass level of these states are given by

$$\frac{m_R^2}{8}=N_R+\frac{1}{2}w^2+h_{in}-\frac{1}{2},$$
$$\frac{m_L^2}{8}=N_L+\frac{1}{2}\bar{w}^2+\frac{1}{2}p^2+\bar{h}_{in}-1, \tag{8.18}$$

where h_{in}, $\bar{h}_{in}$ are the dimensions associated with the internal tensor theory; N_R, N_L are oscillator numbers and p is a vector in the E_8 root lattice.

The generalized GSO projection guarantees space-time SUSY by projecting out all states that does not have odd-integral total superconformal $U(1)$ charge. To impose this condition we introduce the vector

$$\beta_0=(\bar{s}\,;1,\cdots,1\,;1,\cdots,1), \tag{8.19}$$

where $\bar{s}$ is an anti-spinor $SO(2)$ weight. We also define the scalar product

$$V\cdot V'=\frac{1}{4}w\cdot w'-\frac{1}{2}\sum_{i=1}^{r}\frac{q_i\cdot q_i'}{k_i+2}+\frac{1}{4}\sum_{i=1}^{r}s_i\cdot s_i'. \tag{8.20}$$

The total $U(1)$ conformal charge of a state represented by a vector V (including $SO(2)$ or $SO(10)$ contribution) is given by

$$Q_{tot}(V)=2\beta_0\cdot V. \tag{8.21}$$

Notice that Q_{tot} and total Q are related by $Q_{tot}(V)=Q(V)+2w\cdot\bar{s}$. The generalized GSO projection imposes $Q_{tot}=$odd integer. Since the fermions in the various sectors have aligned boundary conditions (either NS or R), another type of projection is also required. Introducing the new vector β_i by

$$\beta_i=(V\,;0,\cdots,0\,;0,\cdots,0),\ (2\text{ in the }i\text{'th position}), \tag{8.22}$$

where V is an $SO(2)$ vector weight, this is achieved by imposing

$$Q_i(V)\equiv 2\beta_i\cdot V=\text{even integer}. \tag{8.23}$$

The above two kinds of projections are achieved independently on the right and left states expressed by V_R and V_L. These two vectors are constrained to differ by an element of the lattice generated by β_0 and the β_i

$$V_L-V_R=m_0\beta_0+m_i\beta_i. \tag{8.24}$$

The character of the right mover $\chi_I^{Het,R}$ is exactly the same as that of SST. But the character of the left mover are given by the level 1 Kac–Moody character $\chi_a^G,(a=O, V, s, \bar{s})$ of the gauge group $G(=SO(26)$, or $SO(10)\times E_8)$. In the case where G is a simply laced Lie algebra a is any weight vector of the algebra G and the characters are given by

$$\chi_a^G = \frac{\Theta_a(\tau, z, u)}{\eta(\tau)^l}, \tag{8.25}$$

where l is the rank of G and the theta functions are at level $k=1$. The level k classical theta functions associated with a Lie algebra G are defined [39, 40]

$$\Theta_a(\tau, z, u) = e^{-2\pi iku} \sum_{\gamma\in M+\frac{a}{k}} e^{\pi ik\tau\gamma^2 - 2\pi ik\gamma z},$$

where a is an element of the weight lattice and M is the lattice spanned by the long roots of the algebra. The χ_a^G's form a unitary representation of the modular group [5]. Then the character $\chi_I^{Het,L}$ is given by

$$\chi_I^{Het,L} = \chi_O^G \chi_I^{(s=0)} + \chi_V^G \chi_I^{(s=2)} + \chi_s^G \chi_I^{(s=1)} + \chi_{\bar{s}}^G \chi_I^{(s=3)}. \tag{8.26}$$

The modular transformation properties are exactly the same as those of χ_I^{SST}. Thus we obtain the modular invariant partition function by

$$Z = \sum_{I,\bar{I}} N_{I,\bar{I}} \chi_I^{Het,R} \chi_{\bar{I}}^{Het,L*}. \tag{8.27}$$

Now we could obtain the heterotic string compactified on Calabi–Yau manifold. Moreover the effective action of the Yukawa couplings are, in principle, exactly calculated by this method [33].

The heterotic compactifications on the K_3 surface have been studied in the context of the representation theory of the $N=4$ conformal field theories [37, 38] with $c=6$. Because there has been no general method to construct the representation wtih $c>3$, it should be emphasized that a lot of studies are remained to be done. Recently it was demonstrated by Kazama and Suzuki that a large class of unitary representations of the $N=2$ superconformal algebra for $c>3$ may be obtained from a generalization of coset construction [38]. Finally throughout this paper we implicitly used the mathematical results in refs. [39, 40].

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Appendix A. Derivation of Kac determinant Formula

Introducing two bosonic and two fermionic oscillators with real parameters λ

and μ

$$\varphi(z)=q_0-i(p_0-\lambda)\ln z+i\sum_{n\neq 0}\frac{a_n}{n}z^{-n}, \tag{A.1a}$$

$$\phi(z)=Q_0-i(P_0-\mu)\ln z+i\sum_{n\neq 0}\frac{c_n}{n}z^{-n}, \tag{A.1b}$$

$$\gamma^i(z)=\sum_n b_n^i z^{-n-\frac{1}{2}},(i=1,2), \tag{A.1c}$$

we represent the currents in the following generic form:

$$T(z)=\frac{1}{2}:A^2(z):+\frac{1}{2}:C^2(z):-\lambda\partial_z A(z)-\mu\partial_z C(z)$$
$$-\frac{1}{2}:\gamma^i(z)\partial_z\gamma^i(z):-\frac{1}{8z}:\gamma^i(z)\gamma^i(z):, \tag{A.2a}$$

$$G^i(z)=\gamma^i(z)A(z)-2\lambda\partial_z\gamma^i(z)\varepsilon^{ij}[\gamma^i(z)C(z)-2\mu\partial_z\gamma^i(z)], \tag{A.2b}$$

$$J(z)=-\frac{i}{2}\varepsilon^{ij}:\gamma^i(z)\gamma^j(z):-2i\lambda C(z)+2i\mu A(z), \tag{A.2c}$$

where $A(z)=i\partial_z\varphi(z)$, $C(z)=i\partial_z\phi(z)$.

In eq.(A.1c) the suffix n runs over $\mathbf{Z}+\frac{1}{2}$ for the A (Neveu-Schwarz) algebra or over $\mathbf{Z}$ for the P (Ramond) algebra. For the T (Twisted) algebra, the suffix n runs over $\mathbf{Z}$ in eq.(A.1c) and over $\mathbf{Z}+\frac{1}{2}$ in eq.(A.1b).

The components of the currents (A.2) satisfy the algebra (2.3) with central charge

$$c=3-12(\lambda^2+\mu^2). \tag{A.3}$$

In the following we can take $\mu=0$ for simplicity, which does not spoil the generality of the discussion.

The vertex operators which correspond to a primary fields multiplet are defined as

$$V(t,u,z)=:\exp[it\varphi(z)+iu\phi(z)]:, \tag{A.4a}$$

$$V^{\pm}(t,u,z)=:(t\pm iu)\gamma^{\pm}(z)V(t,u,z):, \tag{A.4b}$$

$$\mathcal{V}(t,u,z)=:i[\frac{t^2+u^2}{2}\ \varepsilon^{ij}\gamma^i(z)\gamma^j(z)-tC(z)+uA(z)]V(t,u,z):, \tag{A.4c}$$

where $\gamma^{\pm}(z)=\frac{1}{\sqrt{2}}(\gamma^1(z)\pm i\gamma^2(z))$. They form an $N=2$ superconformal multiplet when we put $\mu=0$. The conformal weights of $V, V^{\pm}, \mathcal{V}$ are $h_0, h_0+\frac{1}{2}, h_0+1$, respectively, wher $h_0=\frac{t^2+u^2}{2}+\lambda t$.

The ket vacuum $|\lambda,0\rangle$ is defined by

$$a_n|\lambda,0\rangle=c_n|\lambda,0\rangle=b_n^i|\lambda,0\rangle=0,\ (n>0,i=1,2), \tag{A.5a}$$

$$a_0|\lambda,0\rangle=\lambda|\lambda,0\rangle,\ c_0|\lambda,0\rangle=0. \tag{A.5b}$$

This vacuum satisfies the requirement of superconformal invariance

$$L_n|\lambda, 0\rangle = 0, \quad n \geq -1, \tag{A.6a}$$

$$G_n^i|\lambda, 0\rangle = 0, \quad n \geq -\frac{1}{2}, (i=1,2), \tag{A.6b}$$

$$J_n|\lambda, 0\rangle = 0, \quad n \geq 0. \tag{A.6c}$$

While the ground state $|t+\lambda, u\rangle$ with level$=0$ and $a_0(c_0)$ momentum$=t+\lambda(u)$ satisfies the primary state conditions

$$L_n|t+\lambda, u\rangle = 0, \quad n \geq 1, \tag{A.7a}$$

$$G_n^i|t+\lambda, u\rangle = 0, \quad n \geq \frac{1}{2}, \tag{A.7b}$$

$$J_n|t+\lambda, u\rangle = 0, \quad n \geq 1, \tag{A.7c}$$

$$L_0|t+\lambda, u\rangle = h|t+\lambda, u\rangle, \tag{A.7d}$$

$$J_0|t+\lambda, u\rangle = q|t+\lambda, u\rangle, \tag{A.7e}$$

where $h=\frac{t^2+u^2}{2}+\lambda t, q=-2i\lambda u$ for the sector, $h=\frac{t^2+u^2}{2}+\lambda t+\frac{1}{8}$, $q=\pm\frac{1}{2}-2i\lambda u$ for the R sector and $h=\frac{t^2}{2}+\lambda t+\frac{1}{8}$, $q=0$ for the T sector.

Now consider three special vertex operators $V(-2\lambda, 0, z), V^{\pm}(\frac{1}{2\lambda}, \mp\frac{i}{2\lambda}, z)$. They are conformally invariant vertex operators satisfying the following commutation relations,

$$[L_n, V(-2\lambda, 0, z)] = \partial_z(z^{n+1}V(-2\lambda, 0, z)), \tag{A.8a}$$

$$[G_n^{\pm}, V(-2\lambda, 0, z)] = \pm\partial_z(z^{n+1}V^{\pm}(-2\lambda, 0, z)), \tag{A.8b}$$

$$[J_n, V(-2\lambda, 0, z)] = 0, \tag{A.8c}$$

and

$$[L_n, V^{\pm}(\frac{1}{2\lambda}, \mp\frac{i}{2\lambda}, z)] = \partial_z(z^{n+1}V^{\pm}(\frac{1}{2\lambda}, \mp\frac{i}{2\lambda}, z)), \tag{A.9a}$$

$$\{G_n^{\pm}, V^{\pm}(\frac{1}{2\lambda}, \mp\frac{i}{2\lambda}, z)\} = 0, \tag{A.9b}$$

$$\{G_n^{\pm}, V^{\mp}(\frac{1}{2\lambda}, \mp\frac{i}{2\lambda}, z)\} = \partial_z(z^{n+1}V(\frac{1}{2\lambda}, \mp\frac{i}{2\lambda}, z)), \tag{A.9c}$$

$$[J_n, V^{\pm}(\frac{1}{2\lambda}, \mp\frac{i}{2\lambda}, z)] = 0. \tag{A.9d}$$

Utilizing these conformally invariant vertex operators $V, V^{\pm}$ with conformal weight 1, we can construct singular vertex operators (SVO) for null field as follows:

$$\Phi_r(t, u, z) = \frac{1}{2\pi i}\oint_{c_r} dz_r V(-2\lambda, 0, z_r)\int_z^{z_r} dz_{r-1} V(-2\lambda, 0, z_{r-1})\cdots$$
$$\cdots\int_z^{z_2} dz_1 V(-2\lambda, 0, z_1) V(t, u, z), \tag{A.10a}$$

$$\Psi^{\pm}(t, u, z)=\frac{1}{2\pi i}\oint_{c_r} d\zeta V^{\pm}(\frac{1}{2\lambda}, \mp\frac{i}{2\lambda}, \zeta)V(t, u, z), \qquad (\text{A. 10 b})$$

where the integration contour c_r encicles all the points $z, z_1, \cdots, z_r$.

These SVO's have important properties:

(a) $\Phi_r(t, u, z)$ and $\Psi^{\pm}(t, u, z)$ have the same commutation relations with the L_n, G_n^i, J_n's as $V(t, u, z)$.

(b) $\Phi_r(t, u, z)$ exists nontrivially if and only if t takes one of the values

$$t=t(-r, \frac{s}{2})=\frac{1}{2}(1-r)t_- - \frac{s}{2t_-}, \qquad (\text{A. 11})$$

where r takes positive integers, s takes even integers and $t_-=-2\lambda$ is the solution of the equation $h_0+1=1$.

(c) $\Psi^{\pm}(t, u, z)$ exists nontrivially if and only if t and u satisfy the relation

$$t=t(k)=t_-(k+\frac{1}{2})\pm iu, \qquad (\text{A. 12})$$

where $0<k\in\mathbf{Z}+\frac{1}{2}$.

The null state for the NS sector can be obtained by operating the null field to the state $|\lambda, u\rangle$ which is obtained from the ket vacuum by shifting the c_0 momentum by u:

$$|\chi\rangle=\Phi_r(t, u, z)|\lambda, u\rangle=\Omega|rt_-+t+\lambda, u\rangle. \qquad (\text{A. 13})$$

Here Ω consists of L_{-n}, G_{-n}^i and/or J_{-n}, and the state $|rt_-+t+\lambda, u\rangle$ is the highest weight vector. From the above mentioned property (b) the highest weight vector has a degenerate state (null state) if and only if its momentum $rt_-+t=t(r, \frac{s}{2})$. Then this highest weight vector has the conformal weight

$$h^{NS}(r, \frac{s}{2})=\frac{1}{24}(3-c)(r^2-1)-\frac{1}{4}rs+[\frac{2}{3}(3-c)]^{-1}(\frac{1}{4}s^2-q^2), (s\,;\text{even}), \qquad (\text{A. 14})$$

where $q=-2i\lambda u$ is the eigenvalue of J_0. From (A. 12), we obtain a series of degenerate highest weight:

$$h^{NS\pm}(k)=\pm qk+\frac{1}{6}(3-c)(k^2-\frac{1}{4}). \qquad (\text{A. 15})$$

With eqs. (A. 14) and (A. 15), we can obtain the Kac determinant formula for the NS sector (3. 5).

Similarly, we can obtain the degenerate highest weights for the R sector:

$$h^{R}(r, \frac{s}{2})=\frac{1}{24}(3-c)(r^2-1)-\frac{1}{4}rs+[\frac{2}{3}(3-c)]^{-1}(\frac{1}{4}s^2-q^2)+\frac{1}{8}, (s\,;\text{even}), \qquad (\text{A. 16})$$

and

$$h^{R\pm}(k)=\pm qk+\frac{1}{6}(3-c)(k^2-\frac{1}{4})+\frac{1}{8}, \qquad (\text{A. 17})$$

where $0 \leq k \in \mathbf{Z}$. q is the eigenvalue of J_0 which is given by $q = \pm\frac{1}{2} - 2i\lambda u$ depending on which subspace of the R sector is considered.

With eqs. (A.16) and (A.17), we can obtain the Kac determinant formula for the R sector (3.7).

Finally, for the T sector we have only one series of the highest weights corresponding to the SVO Φ_r. There are no $\Psi^\pm$ type SVO since c_0 momentum is absent for the T sector. Hence we obtain

$$h^T(r, \frac{s}{2}) = \frac{1}{24}(3-c)(r^2-1) - \frac{1}{4}rs + \frac{3}{8}(3-c)^{-1}s^2 + \frac{1}{8}, \ (s\,;\text{odd}), \qquad (\text{A.}18)$$

where $0 < r \in \mathbf{Z}, 0 < \frac{s}{2} \in \mathbf{Z} + \frac{1}{2}$.

With eq. (A.18), we can obtain the Kac determinant formula for the T sector (3.9).

Appendix B. Parafermionic construction of the Character formula

In refs. [18, 19, 20], it has been shown that all the unitary $N=2$ superconformal field theories with $c<3$ are realized as a direct product of the Z_k theory and a $c=1$ theory. The Z_k theory is characterized by a pair of parafermion fields $\psi_1(z)$ and $\psi_1^\dagger(z)$ with fractional opin. The operator product expansion of parafermion currents are given as

$$\psi_1(z)\psi_1^\dagger(w) \sim (z-w)^{-2h_1}[1 + (z-w)^2 \frac{2h_1}{c_\psi} T_k(w) + \ldots], \qquad (\text{B.}1)$$

where T_k is the stress-energy tensor of Z_k theory, $h_1 = \frac{k-1}{k}$ is the dimension of $\psi_1, \psi_1^\dagger$ and c_ψ is the conformal anomaly given by $c_\psi = \frac{2(k-1)}{k+2}$. The relation between the $N=2$ superconformal algebra and the Z_k parafermions is expressed by,

$$G^+(z) = \sqrt{\frac{2k}{k+2}}\psi_1 \exp(i\alpha\phi(z)),$$

$$G^-(z) = \sqrt{\frac{2k}{k+2}}\psi_1^\dagger(z)\exp(-i\alpha\phi(z)), \qquad (\text{B.}2)$$

$$J(z) = \frac{i}{\alpha}\partial_z\phi(z),$$

where ϕ is a free scalar field and $\alpha = \sqrt{(k+2)/k}$. The stress-energy tensor is now given by

$$T(z) = -\frac{1}{2}(\partial_z\phi(z))^2 + T_k(z). \qquad (\text{B.}3)$$

It is straight forward to verify that they satisfy the OPE (2.1) with c given by

$$c = 3[1 - \frac{2}{k+2}], (k=2, 3)\ldots. \qquad (\text{B.}4)$$

A primary field for the $N=2$ superconformal theory will be a product of a parafermionic primary field and a bosonic exponential. For each parafermionic primary field

$$\Phi_q^l : e^{i\alpha_q\phi} :, \tag{B.5}$$

where

$$\alpha_q=\frac{q-(\frac{1}{2}\,\mathrm{sgn}(0)+a)k}{\sqrt{k(k+2)}},\ (q=\dots,l-2,l),$$

$$\alpha_q=\frac{q-(\frac{1}{2}\,\mathrm{sgn}(0)+1+a)k}{\sqrt{k(k+2)}},\ (q=l,l+2,\cdots), \tag{B.6}$$

where in the Ramond sector $a=0$ and $\mathrm{sgn}(0)=\pm 1$ ane in the NS sector $a=\frac{1}{2}$ and $\mathrm{sgn}(0)=-1$. The OPE of $\Phi_q:\exp(i\alpha\phi):$ with $G^{\pm}$ and J can be calculated as:

$$G^{+}(z)\Phi_q^l(0):\exp(i\alpha_q\phi(0)):\sim\sqrt{\frac{2k}{k+2}}z^{\alpha_q\alpha}z^{-\frac{q}{k}}\Phi_{q+2}^k(0):\exp(i(\alpha_q+\alpha)\phi(0)):,$$

$$G^{-}(z)\Phi_q^l(0):\exp(i\alpha_q\phi(0)):\sim\sqrt{\frac{2k}{k+2}}z^{-\alpha_q\alpha}z^{-\frac{k-q}{k}}\Phi_{q-2}^k(0):\exp(i(\alpha_q-\alpha)\phi(0)):,$$

$$J(z)\Phi_q^l(0):\exp(i\alpha_q\phi(0)):\sim\left(\frac{\alpha_q}{2\alpha}\right)z^{-1}\Phi_q^k(0):\exp(i\alpha_q\phi(0)):. \tag{B.7}$$

The conformal dimensions and $U(1)$ charges are then given by,

$$(h,Q)=(h_q+\frac{1}{2}\alpha_q^2,\ \alpha_q/2\alpha), \tag{B.8}$$

where h_q is the dimension of the field Φ_q^l given by

$$h_q=\frac{l(l+2)}{4(k+2)}-\frac{q^2}{4k},$$

for $0\leq l\leq k$, $-k+1\leq q\leq k$ and $l-m=0 \bmod 2$.

Non-primary fields are obtained by acting $G^{\pm}$ and J on the primary fields, in the operator product sense. Thus a superconformal block would contain all field obtained from some primary fields. Operating G^{+} on the field $\Phi_q^l:\exp(i\alpha\phi):$ we get in the leading order the field $\Phi_q^l:\exp(i(\alpha_q+\sqrt{(k+2)/k})\phi):$ as seen from eq. (B.7). Let us assume that k is odd, at first. Then applying G^{+} $2k$ times, the parafermion ψ_1 cancels and we are left with a net shift in the bosonic momenta. This shows that the bosonic piece is given by a theta function at level $2k(k+2)$. Let us separate NS and R sectors into four sectors labeled by s which is defined modulo 4. $s=0,2$ compose the NS sector, and $s=1,3$ compose the R sector. Denote the partition function of the s sector by $\chi_{l,q}^{(s)}(\tau,z)$. Now we can derive the character formula corresponding to the eq.(4.9) as follows;

$$\chi_{l,q}^{(s)}=\sum_{j \bmod k} c_{l,q+4j-s}(\tau)\Theta_{2q+(4j-s)(k+2),2k(k+2)}(\tau,kz,0), \tag{B.9}$$

where in the R sector we have replaced $q \rightarrow q+s$. $\chi_{l,q}^{(s)}$ is invariant under any of the transformations, $s \rightarrow s+4$, $q \rightarrow q+2(k+2)$, which shows that q is defined modulo $2k(k+2)$ and s modulo 4. Also $\chi_{l,q}^{(s)}=0$ if $l+q+s \neq 0$ mod 2. In additon, $\chi_{l,q}^{(s)}$ is invariant under the simultaneous interchange: $l \rightarrow k-l$, $q \rightarrow q+k+2$, and $s \rightarrow s+2$. The result (B.9) is also valid for even k.

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