

Induced Forests in Trees and Outerplanar Graphs

by

Kiyoshi HOSONO

Department of Mathematics, Faculty of Science, Tokai University

(Received on Aug. 15, 1989)

Abstract

We prove the following theorem: for every tree T with n vertices and for every positive integer k , there is a set S of at most $n/(k+1)$ vertices in T such that every component of $T-S$ has most k vertices. We also prove that every outerplanar graph with n vertices contains an induced tree with at least $2n/3$ vertices.

§ 1. Definitions and notation

We deal only with finite, simple graphs without loops or multiple edges.

Akiyama [1] defined the k -tree stability number $\alpha(G; T_k)$ of a graph G as the largest size of a set S of vertices in G such that each component of the subgraph of G induced by S is a tree of order at most k .

Obviously, if G has n vertices then

$$\alpha(G; T_1) \leq \alpha(G; T_2) \leq \cdots \leq \alpha(G; T_n) = \alpha(G; T_{n+1}) = \cdots;$$

we write $\alpha(G; T_\infty) = \alpha(G; T_n)$.

§ 2. Trees

Theorem 2.1

If T is a tree of order n then $\alpha(T; T_k) \geq \lceil kn/(k+1) \rceil$ for all k . Furthermore, this bound is best possible.

(We may assume that $k < n$, for otherwise $\lceil kn/(k+1) \rceil = n$.)

Proof. For each fixed k , we shall use induction on n . If $n \leq k$ then $\lceil kn/(k+1) \rceil = n$, and so the conclusion is trivial. Now we may assume that $n > k$. Call a tree *small* if it has at most k vertices; otherwise call it *big*. We only need to find a vertex v such that the total number of vertices in small components of $T-v$ is at least k : if F denotes the forest consisting of all the big components of $T-v$ then, by the induction hypothesis,

$$\alpha(F; T_k) \geq \lceil k \cdot |V(F)| / (k+1) \rceil \text{ and so}$$

$$\alpha(T; T_k) \geq (n-1-|V(F)|) + \lceil k \cdot |V(F)| / (k+1) \rceil;$$

since $|V(F)| \leq n-1-k$, it follows that $\alpha(T; T_k) \geq \lceil kn / (k+1) \rceil$.

A vertex v with this property can be found by the following algorithm.

Each iteration begins with some vertex w and some big component T' of $T-w$. (To initialize, choose w arbitrarily. If $T-w$ has no big component then set $v=w$ and stop; else let T' be any big component of $T-w$.) Let w' be the unique neighbour of w in T' . If $T'-w'$ has no big component then set $v=w$ and stop; else replace w by w' , replace T' by any big component of $T'-w'$ and proceed to the next iteration.

To see that the bound is best possible, consider the path with n vertices. ■

§ 3. Outerplanar graphs

Theorem 3.1

If G is an outerplanar graph of order n
then $\alpha(G; T_\infty) \geq \lceil 2n/3 \rceil$. Furthermore, this bound is best possible.

Proof. Without loss of generality we may assume that G is a maximal outerplanar graph. It is well known that the chromatic number of an outerplanar graph G is at most 3. Hence, we can color all vertices in G with three colors 1, 2 and 3. Denote by S_{ij} the subgraph of G induced by vertices with color i and j . It is not difficult to see that each of the graphs S_{12}, S_{23} and S_{31} is a tree. Taking the maximum value of $\{|S_{12}|, |S_{23}|, |S_{31}|\}$, a lower bound $\lceil 2n/3 \rceil$ for $\alpha(G; T_\infty)$ is obtained as desired.

To see that the bound obtained above is sharp, consider the series of outerplanar graphs K illustrated in *Figure 1*.

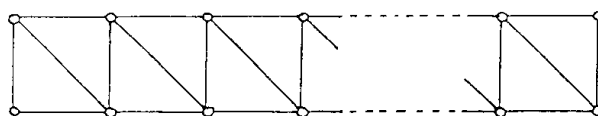


Fig. 1 The series of outerplanar graphs K

Corollary 3.2

If G is an outerplanar graph of order n
then $\alpha(G; T_k) \geq \lceil k \lceil 2n/3 \rceil / (k+1) \rceil$ for all k .

Proof. Immediate from *Theorem 2.1* and *Theorem 3.1*. ■

Acknowledgment

The author is indebted to Prof. V. Chvátal and Prof. H. Enomoto for their helpful comments.

Reference

- [1] Akiyama, J., Era, H., Gervacio, S. V., and Watanabe, M.: Path chromatic numbers of graphs. J. Graph Theory 13 (1989), 569-575.