A Remark on Certain Nonlinear Elliptic Equations

——Dedicated to the memory of Professor M. Fukawa——

by

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1. Introduction

In connection with the maximizing problem for the functional $R(u)=|u|_{L^q}/|u_x|_{L^p}$ in $W_0^{1,p}(0,a)$, Ôtani [5, 6] studied the following one dimensional nonlinear elliptic equation :

$$(E)_a \begin{cases} (1) & (|u_x|^{p-2}u_x)_x+|u|^{q-2}u(x)=0, \quad x\in(0,a), \\ (2) & u(0)=u(a)=0. \end{cases}$$

It is shown in [5, 6] that if p, $q\in(1,+\infty)$ and $p\neq q$, then $(E)_a$ has always exactly countable number of solutions $\{\pm u_k\}_{k\in N}$, which are characterized as unique (modulo $\pm$) solutions possessing $(k-1)$ zeros in $(0,a)$. Here and henceforth, the statement that a function u is a solution of $(E)_a$ means that $u\in W_0^{1,p}(0,a)$ and u satisfies (1) in the distribution sense.

The purpose of the present paper is to examine what happens for the critical case $1<p=q<+\infty$. Recalling the well-known case $p=q=2$, one can easily imagine that the situation is utterly different from that for the case $p\neq q$, for example, the existence of non-trivial solutions may depend on the length of the interval $[0,a]$. In fact, our main result is stated as follows.

Theorem 1. *Let S be the set of all solutions of $(E)_a$ with $1<p=q<+\infty$. Then there exist a positive number a_p and functions $u_n\in$*

$C^1([0,\ na_p])\cap C^2([0,na_p]\setminus\bigcup_{k=1}^{n}\{(2k-1)a_p/2\})$, $n\in N$ *such taht*

(*i*) *If $a=na_p$, then $S=\{\lambda u_n\}_{\lambda\in R^1}$,*

(*ii*) *If $a\neq na_p$, then $S=\{0\}$.*

Moreover $u_1(x)$ is positive in $(0,a)$ and symmetric in x, i.e., $u_1(x)=u_1(a_p-x)$ for $x\in(0,a_p)$ and u_n is given by $u_n(x)=(-1)^k u_1(x-ka_p)$ if $x\in[ka_p,(k+1)a_p]$, $k=0$, 1, ……, $n-1$.

Remark 2.

(i) The exponent a_p has an expression in terms of the beta function as follows : $a_p=(2(p-1)^{1/p}/p)B(1/p, 1/p+1)$.

(ii) u_n are not twice differentiable at $x=(2k-1)a_p/2$ $(k=1, \cdots\cdots, n)$ for the case $2<p$, and u_n belong to $C^2([0, na_p])$ for the case $1<p\leq 2$.

2. Proof of the theorem

We are going to prove the theorem by several steps. The following result is shown in [5, 6].

Lemma 3. (Regularity of solutions) *Let u be a non-trivial solution of $(E)_a$. Then we have :*

(i) *If $1<p\leq 2$, then $u\in C^2([0, a])$,*

(ii) *If $2<p$, then $u\in C^1([0, a])\cap C^2([0, a]\backslash Z)$ and*
u is not twice differentiable at $Z=\{x\in[0, a]\ ;\ u_x(x)=0\}$.

Since $W_0^{1,p}(0, a)$ is compactly embedded in $L^p(0, a)$, we can well define $C_p(a)$ by

(3) $C_p(a)=\max\{R(u)\ ;\ u\in W_0^{1,p}(0, a),\ u\neq 0\}$,

where $R(u)=|u|_{L^p}/|u_x|_{L^p}$.

In other words, $C_p(a)$ gives the best possible constant for the Sobolev-Pointcaré-type inequality :

(4) $|u|_{L^p}\leq C|u_x|_{L^p}$ for $u\in W_0^{1,p}(0, a)$.

The a-dependance of $C_p(a)$ is given explicitly by the following lemma.

Lemma 4. *We have $C_p(a)=aC_p(1)$ for all $a>0$.*

Proof. Let us introduce two operators $T_a : W_0^{1,p}(0, a)\to W_0^{1,p}(0, 1)$ and $T_{1/a} : W_0^{1,p}(0, 1)\to W_0^{1,p}(0, a)$ by

$(T_aw)(x)=w(ax)$ $x\in[0, 1]$,

$(T_{1/a}w)(x)=w(x/a)$ $x\in[0, a]$.

Then simple calculation shows that

(5) $R(T_bw)=R(w)/b$

holds for $b=a$ and $1/a$, where $R(v)$ is the ratio $|v|_{L^p}/|v_x|_{L^p}$ on the interval where v is defined.

Hence, taking two functions w_a and w_1 which maximize the functional $R(w)$ on $W_0^{1,p}(0, a)\backslash\{0\}$ and $W_0^{1,p}(0, 1)\backslash\{0\}$ respectively, we find by (5) that

$C_p(a)=R(w_a)=aR(T_aw_a)\leq aC_p(1)$

and

$C_p(1)=R(w_1)=(1/a)R(T_{1/a}w_1)\leq(1/a)C_p(a)$,

whence follows the desired relation. Q. E. D.

As a direct consequence of this lemma, we have the following result.

Corollary 4. *Let $a_p=1/C_p(1)$ and $a\in(0,a_p)$, then $(E)_a$ has no non-trivial solution.*

Proof. Let u be a solution of $(E)_a$. Then multiplication of (1) by u and integration by parts on $(0,a)$ give

(6) $|u|_{L^p}=|u_x|_{L^p}$.

On the other hand, recalling that $C_p(a)$ is the best possible constant for (4), we get

(7) $|u|_{L^p}\leq C_p(a)|u_x|_{L^p}$.

Here, from Lemma 4, $C_p(a)<1$ if and only if $a\in(0,a_p)$. Then (6) and (7) imply that $u=0$. Q. E. D.

In showing the existence of non-trivial solutions for some nonlinear elliptic equations, the following abstract result is often useful.

Lemma 5. *Let $\partial\phi^i(i=1,2)$ be subdifferentials of lower semicontinuous convex functions ϕ^i from a real Hilbert space H into $(-\infty,+\infty]$ with $\phi^i\not\equiv+\infty$ such that $D(\phi^1)\subset D(\phi^2)$. Suppose that there exists an element $u\in D(\partial\phi^2)$ such that $J(u)=\min\{J(v)\,;\,v\in D(\phi^1)\}$, where $J(u)=\phi^1(u)-\phi^2(u)$. Then $u\in D(\partial\phi^1)$ and $\partial\phi^2(u)\subset\partial\phi^1(u)$, i. e., $\partial\phi^1(u)=\partial\phi^2(u)$ for the case that $\partial\phi^i$ are single-valued.*

Proof. Since u minimize $J(v)$ in $D(\phi^1)$, we get

$\phi^1(v)-\phi^1(u)\geq\phi^2(v)-\phi^2(u)$ for all $v\in D(\phi^1)$.

Hence, from the definition of the subdifferntial, we find

$\phi^1(v)-\phi^1(u)\geq\phi^2(v)-\phi^2(u)\geq(g,v-u)_H$

holds for all $v\in D(\phi^1)$ and $g\in\partial\phi^2(u)$, which implies that

$u\in D(\partial\phi^1)$ and $g\in\partial\phi^1(u)$. Q. E. D.

Corollary 5. *Let $a=a_p$, then $(E)_a$ has a non-trivial solution.*

Proof. Put

$$\phi^1(u)=\begin{cases}\frac{1}{p}\int_0^{a_p}|u_x(x)|^p dx & \text{if } u\in W_0^{1,p}(I_p),\\ +\infty & \text{if } u\in L^2(I_p)\setminus W_0^{1,p}(I_p),\end{cases}$$

and

$$\phi^2(u)=\begin{cases}\frac{1}{p}\int_0^{a_p}|u(x)|^p dx & \text{if } u\in L^2(I_p)\cap L^p(I_p),\\ +\infty & \text{if } u\in L^2(I_p)\setminus L^p(I_p),\end{cases}$$

where $I_p=(0,a_p)$. Then ϕ^i become lower semi-continuous convex functions on $H=L^2(I_p)$, and we have $D(\phi^1)=W_0^{1,p}(I_p)$, $D(\phi^2)=L^2(I_p)\cap L^p(I_p)$, $D(\partial\phi^2)=L^{2(p-1)}$

$(I_p)\cap L^2(I_p)$, $\partial\phi^1(u)=-(|u|_x^{p-2}u_x)_x$ and $\partial\phi^2(u)=|u|^{p-2}u$. Moreover, since the best possible constant in (4) is $C_p(a_p)=1$, there exists a function $u\in W_0^{1,p}(I_p)\setminus\{0\}$ such that $J(u)=0=\min\{J(v)\ ;\ v\in W_0^{1,p}(I_p)\}$. Thus, by virtue of Lemma 5, u turns out to be a non-trivial solution of $(E)_a$. Q. E. D.

The structure of the set of all solutions of $(E)_a$ is determined as follows.

Lemma 6. *Let $a=a_p$, then $(E)_a$ has a unique positive solution $u_1(x)$ satisfying $(u_1)_x(0)=1$ and the set of all solutions of $(E)_a$ consits of $\{\lambda u_1\}_{\lambda\in R^1}$. Moreover $u_1(x)$ is symmetric in x, i.e., $u_1(x)=u_1(a_p-x)$ for all $x\in(0,a_p)$.*

Proof. Let v be a solution of $(E)_a$, then we get

(8) $|v_x|^{p-2}v_x(x)=-\int_0^x|v|^{p-2}v(t)dt+\text{Const.}$ for all $x\in(0,a)$,

since $(|v_x|^{p-2}v_x,\phi_x)=\left(-\int_0^x|v|^{p-2}v(t)dt,\ \phi_x\right)$ for all $\phi\in C_0^\infty(0,a)$.

Furthermore, multiplying (1) by $u_x(x)$ and using the equality $|v|^{r-2}v(x)v_x(x)=(|v|^r)_x(x)/r$ for $r>1$, we obtain

(9) $\left(\frac{p-1}{p}|v_x(x)|^p+\frac{1}{p}|v(x)|^p\right)_x=0$ for all $x\in(0,a)\setminus Z$,

since $u\in C^2([0,a]\setminus Z)$ by Lemma 3 and $(|v_x|^{p-2}v_x)_xv_x=(p-1)|v_x|^{p-2}v_xv_{xx}$.

Let u be any non-trivial solution of $(E)_{a_p}$. Suppose that $u(a)=0$ for some $a\in(0,a_p)$, then it is easy to see that $u(x)$ or $u(a_p-x)$ gives a non-trivial solution of $(E)_a$ or $(E)_{a_p-a}$, which contradicts Corollary 4. Thus $u(x)>0$ (or $u(x)<0$) for all $x\in(0,a_p)$. Hence, by(8), $u_x(x)$ is monotone decreasing (or increasing) in x. Therefore $u_x(x)$ attains zero only once in $(0,a_p)$. Then, by virtue of (9) and the fact that $u\in C^1([0,a_p])$, we deduce

(10) $(p-1)|u_x(x)|^p+|u(x)|^p=(p-1)|u_x(0)|^p$ for all $x\in(0,a_p)$.

We further claim that $|u(x)|$ attains its maximum at $x=a_p/2$. In fact, if $|u(x)|$ attains its maximum at $x=b\in(0,a_p/2)$, then we can construct another non-trivial solution v of $(E)_{2b}$ by putting $v(x)=u(x)$ for $x\in(0,b)$ and $v(x)=u(2b-x)$ for $x\in(0,2b)$. This contradicts Corollary 4.

Let u_1 and v_1 be positive solutions of $(E)_{a_p}$ such that $(u_1)_x(0)=(v_1)_x(0)=1$ and $u_1\not\equiv v_1$. Then, from (10), u_1 and v_1 attain the same maximum at $x=a_p/2$. Hence there exists an interval $(a,b)\subset(0,a_p/2)$ such that $u_1(a)=v_1(a)$, $u_1(b)=v_1(b)$, and $u_1(x)>v_1(x)$ for all $x\in(a,b)$. Then (10) gives

(11) $(u_1)_x(a)=(v_1)_x(a)$ and $(u_1)_x(b)=(v_1)_x(b)$,

and integration of (8) on (a,b) gives

$$\begin{aligned}&|(u_1)_x|^{p-2}(u_1)_x(b)-|(u_1)_x|^{p-2}(u_1)_x(a)\\&=-\int_a^b|u_1|^{p-2}u_1(x)\,dx<-\int_a^b|v_1|^{p-2}v_1(x)dx\\&\leq|(v_1)_x|^{p-2}(v_1)_x(b)-|(v_1)_x|^{p-2}(v_1)_x(a),\end{aligned}$$

which contradicts (11). Thus $(E)_{a_p}$ has a unique positive solution u_1 satisfying $(u_1)_x(0)=1$.

Since $v_1(x)=u_1(a_p-x)$ is also a positive solution of $(E)_{a_p}$ with $(v_1)_x(0)=1$ by (10), the uniqueness of u_1 implies that $u_1(x)$ must be symmetric in x.

Furthermore, it follows from (10) that any non-trivial solution u of $(E)_{a_p}$ satisfies $u_x(0)\neq 0$. Then, noting that $v(x)=u(x)/u_x(0)$ gives the positive solution satisfying $v_x(0)=1$, we find that $u(x)=u_x(0)u_1(x)$ for all $x\in(0,a_p)$. Q. E. D.

As for the case $a>a_p$, we have :

Lemma 7. *Let $a>a_p$, then $(E)_a$ has no positive solution.*

Proof. Let u be a positive solution of $(E)_a$, then it is clear that u also satisfies (10) for all $x\in[0,a]$. Hence $v(x)=u(x)/u_x(0)$ gives a positive solution of $(E)_a$ with $v_x(0)=1$ and we may assume that $v(x)$ attains its maximum $m=(p-1)^{1/p}$ only once at $x=b\in[a/2,a]$. (If this is not the case, then $v(2b-x)$ satisfies this property.) Put $w(x)=mv(x)/v(a_p/2)$, then w satisfies $w(a_p/2)=u_1(a_p/2)=m$ and $w_x(0)=m/v(a_p/2)>1=(u_1)_x(0)$, since $v(a_p/2)<m$ by (10). Hence there exists an interval $(0,b)\subset(0,a_p/2)$ such that $w(0)=u_1(0)$, $w(b)=u_1(b)$ and $w(x)>u_1(x)$ for all $x\in(0,b)$. Then, by (8), we get

$$
\begin{aligned}
&|w_x|^{p-2}w_x(b)-|w_x|^{p-2}w_x(0)\\
&=-\int_0^b|w|^{p-2}w(x)dx<-\int_0^b|u_1|^{p-2}u_1(x)dx \qquad (12)\\
&=|(u_1)_x|^{p-2}(u_1)_x(b)-|(u_1)_x|^{p-2}(u_1)_x(0)\\
&=(1-|u_1(b)|^p/(p-1))^{(p-1)/p}-1.
\end{aligned}
$$

Put $F(t)=(t^p-|u_1(b)|^p/(p-1))^{(p-1)/p}-t^{p-1}$, then (12) gives

(13) $F(M)<F(1)$ with $M=w_x(0)>1$.

On the other hand, it is easy to see that $F(t)$ is monotone increasing in $t\geq 1$. This is a contradiction to (13). Q. E. D.

Now we are ready to prove the theorem.

Proof of theorem 1. Let u be a non-trivial solution of $(E)_a$. Then, since u belongs to $C^1([0,a])$ by Lemma 3, there exists a point $x_0\in(0,a)$ such that $|u(x)|$ attains its positive maximum at $x=x_0$. Hence there exists an interval (b,c) containing x_0 such that $|u(x)|>0$ for all $x\in(b,c)$ and $u(b)=u(c)=0$, i.e., $|u(x)|$ is a positive solution of the equation (1) on $[b,c]$. Then it follows from Corollary 4, Lemmas 6 and 7 that $c-b=a_p$ and $u(x)=u_x(b)u_1(x-b)$ for all $x\in[b,c]$. Since $u_x(b)=-u_x(c)\neq 0$, there exists another number $d\in(c,a)$ (if $c\neq a$) such that $|u(x)|$ becomes a positive solution of (1) on $[c,d]$. Then we again deduce that $d-c=a_p$ and $u(x)=-u_x(b)u_1(x-c)$ for all $x\in[c,d]$. Thus, repeating this procedure, we can verify

the assertion (i) and (ii). As for the regularity of u_n, it suffices to recall Lemma 3 and the fact that $\{x\in[0, a_p]\ ;\ (u_1)_x(x)=0\}=\{a_p/2\}$. Q. E. D.

Proof of remark 2. Since u_1 satisfies (10) and $(u_1)_x(x)>0$ for all $x\in(0, a_p/2)$, we have

$$du_1(x)/dx=\{1-u_1(x)^p/(p-1)\}^{1/p} \text{ for all } x\in(0, a_p/2).$$

Then

$$\begin{aligned} a_p&=2\int_0^{a_p/2} dx=2\int_0^{(p-1)^{1/p}}\{1-u^p/(p-1)\}^{-1/p}du \\ &=2(p-1)^{1/p}\int_0^1(1-u^p)^{-1/p}du \\ &=2(p-1)^{1/p}/p\int_0^1 t^{1/p-1}(1-t)^{1/p}dt \\ &=2(p-1)^{1/p}/p\ B(1/p, 1/p+1). \end{aligned}$$

The assertion (ii) is a direct consequence of Lemma 3 and the fact that $\{x\in[0, a_p]\ ;\ (u_1)_x(x)=0\}=\{a_p/2\}$. Q. E. D.

Remark 8. Take another parameter λ instead of a, the lenght of the interval, and consider the following equation $(E)^\lambda$ in the fixed interval $(0, 1)$.

$$(E)^\lambda\begin{cases} -(|u_x|^{p-2}u_x)_x(x)=\lambda|u|^{p-2}u(x), & x\in(0.1), \\ u(0)=u(1)=0. \end{cases}$$

Then Theorem 1 asserts that eigenvalues of $(E)^\lambda$ are given by $\lambda_k=(ka_p)^p$, $k=1, 2, \cdots\cdots$, and eigenvector spaces associated with λ_k are $\{\mu v_k\}_{\mu\in R^1}$, where $v_k(x)=u_k(ka_px)$, $x\in(0, 1)$.

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