A Carleson Measure Theorem for Weighted Bergman Spaces

by

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Abstract

The main purpose of this note is to generalize a theorem of W. W. Hastings [1] : let μ be a finite, positive measure on D, the unit disc in $\mathbb{C}$, and let σ be a Lebesgue measure on D. For $1\leq p\leq q<\infty$, $\alpha>-1$ and $\beta\leq 0$, a necessary and sufficient condition on μ is given in order that $\left\{\int_D (1-|z|)^{\beta q/p} f^p(z)\,d\mu(z)\right\}^{1/q}\leq\left\{\int_D (1-|z|)^{\alpha} f^p(z)\,d\sigma(z)\right\}^{1/p}$ for every positive subharmonic function $f(z)$ on D.

Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$ be analytic in the open unit disc $D: |z|<1$, and let

$$M_p(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{1/p}\ ;\ 0<p<\infty$$

for $0\leq r<1$. Let H^p be the set of function analytic in D such that

$$\|f\|_{H^p}=\sup_{r<1} M_p(r,f)$$

is finite. The weighted Bergman space A^p_α for $0<p<\infty$ and $\alpha>-1$ is the linear space of all functions $f(z)$ analytic in D, for which the integral

$$\|f\|^p_{p,\alpha}=\int_0^1 (1-r)^\alpha M_p^p(r,f)\,dr$$

is finite. It is easily seen that $H^p\subset A^p_\alpha$, $(\alpha>-1)$ and $A^p_\alpha\subset A^p_\beta$, $(-1<\alpha<\beta)$.

A theorem of Carleson generalized by P. L. Duren [2] is as follows :

Theorem A. Let μ be a finite, positive measure on D and $0<p\leq q<\infty$. Then in order that there exists a constant $C>0$ such that

$$\left\{\int_D |f(z)|^q d\mu(z)\right\}^{1/q}\leq C\|f\|_{H^p}$$

for all $f(z)\in H^p$, it is necessary and sufficient that $\mu(S)\leq A\delta^{q/p}$ for every set of S

of the form

$$S=\{z=re^{i\theta}: 1-\delta\leq r<1, \theta_0\leq\theta\leq\theta_0+\delta\}. \tag{1}$$

W. W. Hastings [1] proved this theorem for A_0^p. In fact, his result is more general:

Theorem B. Let μ be a finite, positive measure on D, σ be a Lebesgue measure on D and suppose $1\leq p\leq q<\infty$. Then there exists a constant $C>0$ such that

$$\left\{\int_D f^q(z)d\mu(z)\right\}^{1/q}\leq C\left\{\int_D f^p(z)\,d\sigma(z)\right\}^{1/p}$$

for every positive subharmonic function $f(z)$ on D if and only if there exists a constant $A>0$ such that

$$\mu(S)\leq A\delta^{2q/p} \tag{2}$$

for every set S of the form (1).

As an immediate consequence of this theorem, we can see that the condition (2) is equivalent to

$$\left\{\int_D |f(z)|^q d\mu(z)\right\}^{1/q}\leq C\|f\|_{p,0}$$

for all $f(z)\in A_0^p$, where $0<p\leq q<\infty$.

We will generalize the theorem B for weighted Bergman space A_α^p, $(\alpha>-1)$.

Theorem 1. The condition for μ, σ, p and q are the same as in theorem B. We suppose $\alpha>-1$ and $\beta\leq 0$. Then there exists a constant $C>0$ such that

$$\left\{\int_D (1-|z|)^{\beta q/p} f^q(z)d\mu(z)\right\}^{1/q}\leq C\left\{\int_D (1-|z|)^\alpha f^p(z)d\sigma(z)\right\}^{1/p} \tag{3}$$

for every positive subharmonic function $f(z)$ on D if and only if there exists a constant $A>0$ such that

$$\mu(S)\leq A\delta^{q/p\{2-(\beta-\alpha)\}} \tag{4}$$

for every set of the form (1).

To prove this theorem, three lemmas are needed.

Lemma 1. Let α, β be a real number and set

$$z_{mk}=(1-2^{-m})\exp\{2(k+2^{-1})\pi i/2^{m+4}\},$$

where $k=1, 2, 3, \cdots\cdots$; $m=0, 1, 2, \cdots\cdots$.

Then we have the inequality

$$C_{\alpha,\beta}\leq 2^{m\{2-(\beta-\alpha)\}}\int_{6/2^{m+3}}^{7/2^{m+3}}\min_\theta (1-|z_{mk}+\rho e^{i\theta}|)^{-(\beta-\alpha)}\rho d\rho,$$

where $C_{\alpha,\beta}$ is a positive constant depending only on α and β.

Proof. First let $\beta-\alpha\geq 0$, and $m\geq 1$. Then we have that for every $\theta\in[0, 2\pi]$

$$\int_{6/2^{m+3}}^{7/2^{m+3}}\rho(1-|z_{mk}+\rho e^{i\theta}|^{-(\beta-\alpha)}d\rho$$

$$\geq \frac{3}{2^2} 2^{-m} \int_{6/2^{m+3}}^{7/2^{m+3}} (1-|z_{mk}|+\rho)^{-(\beta-\alpha)} d\rho$$

$$\geq \frac{3}{2^5} \left(\frac{2^3}{15}\right)^{(\beta-\alpha)} 2^{-m\{2-(\beta-\alpha)\}}$$

$$= C^{(1)}_{\alpha,\beta} 2^{-m\{2-(\beta-\alpha)\}}.$$

If we suppose $m=0$, then we have the inequality,

$$\int_{6/2^3}^{7/2^3} \rho(1-|z_{0k}+\rho e^{i\theta}|)^{-(\beta-\alpha)} d\rho$$

$$\geq \frac{3}{2^2} \int_{6/2^3}^{7/2^3} (1-\rho)^{-(\beta-\alpha)} d\rho$$

$$\geq \frac{3}{2^5} 2^{2(\beta-\alpha)}$$

$$\geq C^{(1)}_{\alpha,\beta}.$$

Next suppose that $\beta-\alpha<0$ and $m\geq 0$, then we have

$$\int_{6/2^{m+3}}^{7/2^{m+3}} \rho(1-|z_{mk}+\rho e^{i\theta}|)^{-(\beta-\alpha)} d\rho$$

$$\geq \frac{3}{2^2} 2^{-m} \int_{6/2^{m+3}}^{7/2^{m+3}} (1-|z_{mk}|-\rho)^{-(\beta-\alpha)} d\rho$$

$$\geq \frac{3}{2^5} 2^{3(\beta-\alpha)} 2^{-m\{2-(\beta-\alpha)\}}$$

$$= C^{(2)}_{\alpha,\beta} 2^{-m\{2-(\beta-\alpha)\}}.$$

Thus if we put

$$C_{\alpha,\beta} = \min\,(C^{(1)}_{\alpha,\beta},\ C^{(2)}_{\alpha,\beta})$$

then the desired result is obtained.

Lemma 2. Let

$$T_m = \{z=re^{i\theta} : 1-1/2^m \leq r < 1-1/2^{m+1}, 0\leq\theta\leq 2\pi\}\ ;\ m=0,1,2,\cdots\cdots.$$

Then for $z\in T_m$ and $\rho\in[6/2^{m+3}, 7/2^{m+3}]$, we have the inequality

$$(1-|z|)^\beta \leq C_\beta(1-|z_{mk}+\rho e^{i\theta}|)^\beta \tag{5}$$

for real β and for all $\theta\in[0,2\pi]$.

Proof. It is easy to see that for $z\in T_m$,

$$1-|z|\leq 2^{-m}$$

and

$$1-|z|\geq \frac{1}{2}\,2^{-m}.$$

On the other hand, since

$$1-|z_{mk}+\rho e^{i\theta}|\geq \frac{1}{2^3}\,2^{-m}$$

and

$$1-|z_{mk}+\rho e^{i\theta}|\leq\frac{15}{2^3}2^{-m},$$

we have

$$1-|z|\leq 2^3(1-|z_{mk}+\rho e^{i\theta}|) \tag{6}$$

and

$$\frac{2^2}{15}(1-|z_{mk}+\rho e^{i\theta}|)\leq 1-|z| \tag{7}$$

respectively. Thus if $\beta\geq 0$ by (6), and if $\beta<0$ by (7), we obtain the desired inequality (5).

Lemma 3. (lemma 6 in [3]).
For $\alpha>-1$ and $\gamma>1+\alpha$ we have

$$\int_0^1\frac{(1-r)^\alpha}{(1-\rho r)^\gamma}dr\leq\frac{C}{(1-\rho)^{\gamma-\alpha-1}}\ ;\ 0<\rho<1.$$

Proof of Theorem 1. (sufficiency) We first define that for $m=0,1,2,\cdots\cdots$ and $k=1,2,3,\cdots\cdots,2^{m+4}$,

$$T_{mk}=\{z=re^{i\theta}: 1-2^{-m}\leq r<1-2^{-m-1},\ 2k\pi/2^{m+4}\leq\theta\leq 2(k+1)\pi/2^{m+4}\}.$$

In view of W. W. Hastings [1], if $1\leq p<\infty$, then for $\rho\in[6/2^{m+3},\ 7/2^{m+3}]$ and $z\in T_{mk}$,

$$f^p(z)\leq C\frac{1}{2\pi}\int_0^{2\pi}f^p(z_{mk}+\rho e^{i\theta})d\theta \tag{8}$$

for every positive subharmonic function $f(z)$. Therefore by (8) and lemma 2,

$$\begin{aligned}&(1-|z|)^\beta f^p(z)\\&\leq C_\beta'\frac{1}{2\pi}\int_0^{2\pi}(1-|z_{mk}+\rho e^{i\theta}|)^\beta f^p(z_{mk}+\rho e^{i\theta})d\theta.\end{aligned} \tag{9}$$

Hence by lemma 1 and (9),

$$\begin{aligned}&(1-|z|)^\beta f^p(z)\\&\leq C_{\alpha,\beta}^{-1}2^{m\{2-(\beta-\alpha)\}}\int_{6/2^{m+3}}^{7/2^{m+3}}(1-|z|)^\beta f^p(z)\min_\theta(1-|z_{mk}+\rho e^{i\theta}|)^{-(\beta-\alpha)}\rho d\rho\\&\leq C'_{\alpha,\beta}2^{m\{2-(\beta-\alpha)\}}\frac{1}{2\pi}\int_{6/2^{m+3}}^{7/2^{m+3}}\int_0^{2\pi}(1-|z_{mk}+\rho e^{i\theta}|)^\alpha f^p(z_{mk}+\rho e^{i\theta})\rho d\theta d\rho\\&\leq C'_{\alpha,\beta}2^{m\{2-(\beta-\alpha)\}}\int_{D_{mk}}(1-|w|)^\alpha f^p(w)d\sigma(w)\end{aligned}$$

where, $D_{mk}=\{z\in\mathbb{C}: |z-z_{mk}|<7/2^{m+3}\}$.
While, by the hypothesis (4), we get

$$\mu(T_{mk})\leq C2^{-(mq/p)\{2-(\beta-\alpha)\}}.$$

Therefore

$$\begin{aligned}&\int_D(1-|z|)^{(q\beta/p)}f^p(z)d\mu(z)\\&=\sum_{m=0}^\infty\sum_{k=1}^{2^{m+4}}\int_{T_{mk}}(1-|z|)^{(q\beta/p)}f^q(z)d\mu(z)\end{aligned}$$

$$\leq C_{\alpha,\beta,p,q} \sum_m \sum_k \mu(T_{mk}) 2^{(mq/p)\{2-(\beta-\alpha)\}} \left\{\int_{D_{mk}} (1-|w|)^\alpha f^p(w) d\sigma(w)\right\}^{q/p}$$

$$\leq C_{\alpha,\beta,p,q} \sum_m \sum_k \left\{\int_{D_{mk}} (1-|w|)^\alpha f^p(w) d\sigma(w)\right\}^{q/p}.$$

These inequalities correspond to the one from the sixth to eighth lines of page 239 in [1]. The rest of the proof runs in the same way as in [1], and we can obtain the inequality (3).

(necessity) Suppose that $0<p\leq q<\infty$. Let

$$z_0=(1-\delta)\exp\{i(\theta_0+\delta/2)\}$$

and set

$$f(z)=|1-\bar{z}_0 z|^{-(4+\alpha)/p}.$$

If $\alpha>-1$, then we can see that by lemma 3

$$\begin{aligned}&\int_D (1-|z|)^\alpha f^p(z) d\sigma(z)\\ &\qquad \leq C\int_0^1 \frac{(1-r)^\alpha}{\{1-(1-\delta)r\}^{3+\alpha}} dr\\ &\qquad \leq C'\delta^{-2}.\end{aligned}$$

And if $\beta\leq 0$ and $z\in S$, it follows that

$$\int_D (1-|z|)^{q\beta/p} f^q(z) d\mu(z) \geq C\delta^{-q/p\{4-(\beta-\alpha)\}}\mu(S).$$

Thus if (3) holds, we get the inequality (4). (see [1] for details)

Theorem 2. Suppose $0<p\leq q<\infty$ and $\beta\leq 0$, then we have the equivalence :

$$\left\{\int_D (1-|z|)^{q\beta/p} |f(z)|^q d\mu(z)\right\}^{1/q} \leq C\|f\|_{p,\alpha} \text{ for all } f(z)\in A^p_\alpha$$

$$\Longleftrightarrow \text{ (4) for every set } S \text{ of the form (1).}$$

The proof is the same as theorem 1, since $|f(z)|^p$ is subharmonic in D, if $0<p<\infty$ and $f(z)$ is analytic in D.

Suppose $\{z_k\}$ is a sequence of distinct points in D, and let μ be the point measure, defined by $\mu(\{z_k\})=(1-|z_k|)^{2+\alpha}$, $\mu(D\setminus\{z_k\})=0$.

Corollary 1. If $0<p<\infty$, then we obtain the equivalence :

$$\sum_{k=1}^\infty (1-|z_k|)^{2+\alpha} |f(z_k)|^p<\infty \text{ for } f(z)\in A^p_\alpha$$

$$\Longleftrightarrow \mu(S)\leq C\delta^{2+\alpha} \text{ for every set } S \text{ of the form (1).}$$

Let us define that a sequence $\{z_k\}$ is uniformly separated if there is a number $\delta>0$ such that

$$\prod_{\substack{j=1\\ j\neq k}}^\infty \left|\frac{z_k-z_j}{1-\bar{z}_j z_k}\right| \geq \delta,\ k=1,2,3,\cdots\cdots.$$

Theorem C. (see p. 149–157 in [4]). Let $\{z_k\}$ be uniformly separated. Then there is a constant $C>$ such that for every set of the form (1),

$$\mu(S) \leq C\delta.$$

With the aid of theorem C,

Corollary 2. If $\{z_k\}$ is uniformly separated, then for any $f(z) \in A^p_\alpha$

$$\sum_{k=1}^{\infty} (1-|z_k|)^{2+\alpha} |f(z_k)|^p < \infty.$$

We remark that there is a special case of corollary 2 in [5] and [6] i. e., when $p=2$ and $\alpha=0$, respectively.

We have already conjectured in [7] that if a sequence $\{z_k\}$ is uniformly separated, then for every $f(z) \in B^p$ ($B^p = A^1_{1/p-2}$, see [8] for details) we shall get

$$\sum_{k=1}^{\infty} (1-|z_k|)^{2-p} |f(z_k)|^p < \infty.$$

But we can only obtain the following by the use of corollary 2.

Corollary 3. If $\{z_k\}$ is uniformly separated and $f(z) \in B^p$, then

$$\sum_{k=1}^{\infty} (1-|z_k|)^{1/p} |f(z_k)| < \infty.$$

Finally, we state the two inequalities as in [2].

Corollary 4. If $0 < p \leq q < \infty$, then $f(z) \in A^p_\alpha$ implies

$$\int_0^1 (1-r)^{q/p(2+\alpha)-2} M_q^q(r, f)\, dr < \infty$$

and

$$\int_{-1}^1 (1-r)^{q/p(2+\alpha)-1} |f(r)|^q dr < \infty.$$

The first inequality appears in [9].

Remark and Errata. In addition to theorem B, we remark that he proved the theorem for D^n instead of D. In [7], we proved that $\ell^p \subset T_p(B^p) \Longrightarrow \{z_k\} \in S(U)$ and $\ell^p \subset T_p(B^q) \Longrightarrow \{z_k\} \in S(U)$ held in theorem 1 and theorem 1′ respectively. But since we recognize that there is an error in the proof, we omit the above parts.

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References

[1] W. W. Hastings, A Carleson measure theorem for Bergman spaces, Proc. Amer. Math. Soc. 52 (1975), 237-241.

[2] P. L. Duren, Extension of a theorem of Carleson, Bull. Amer. Math. Soc. 75 (1965), 143-146.

[3] A. L. Shields and D. L. Williams, Bounded projections, duality, and multipliers in spaces of analytic functions, Trans. Amer. Math. Soc. 162 (1971), 287-302.

[4] P. L. Duren, Theory of H^p spaces, Pure and Appl. Math., Vol. 38, Academic Press, New York, 1970.

[5] K. Øyma, Interpolation in H^p-spaces, Proc. Amer. Math. Soc. 76 (1979), 81-88.

[6] E. Amar, Suites d'interpolation pour le classes de Bergman de la boule et du polydisque de $\mathbb{C}^n$, Canad. J. Math. 30 (1978) no. 4, 711-737.

[7] F. Ohya, Notes on interpolation theorems for the class B^p, Proc. Fac. Sci. Tokai Univ. 13 (1977), 11-18.

[8] P. L. Duren, B. W. Romberg and A. L. Shields, Linear functionals on H^p spaces with $0<p<1$, J. Reine Angew. Math. 238 (1969), 32-60.

[9] A. Nakamura, F. Ohya and H. Watanabe, On some properties of functions in weighted Bergman spaces, Proc. Fac. Sci. Tokai Univ. 15 (1979), 33-44.