Note on Fourier-Stieltjes Transforms

by

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Let G be a locally compact abelian group with dual Γ, and $M(G)$ be the Banach algebra of bounded regular Borel measures on G. For a closed subset X of G and a bounded continuous function f on Γ, we consider the following three conditions;

(a) there exists $\mu \in M(G)$ such that supp $\mu \subset X$ and $f = \hat{\mu}$,

(b) for all sequences $\{\lambda_n\}$ in $M(\Gamma)$ such that $\{\sup\limits_{x \in X} |\hat{\lambda}_n(x)|\}$ are bounded and

$$\lim_{n\to\infty} \hat{\lambda}_n(x) = 0 \quad (x \in X), \qquad \lim_{n\to\infty} \int_\Gamma f d\lambda_n = 0$$

holds,

(b′) for all sequences $\{\phi_n\}$ in $L^1(\Gamma)$ such that $\{\sup\limits_{x \in X} |\hat{\phi}_n(x)|\}$ are bounded and

$$\lim_{n\to\infty} \hat{\phi}_n(x) = 0 \quad (x \in X), \qquad \lim_{n\to\infty} \int_\Gamma f \phi_n d\gamma = 0$$

holds.

In 1) it was shown that (a) and (b) are equivalent. We will prove here the following theorem.

Theorem (a) and (b′) are equivalent.

Proof (a)$\Longrightarrow$(b′) By the Fubini's theorem and the Lebesgue's dominated convergence theorem,

$$\int_\Gamma f\phi_n d\gamma = \int_X \hat{\phi}_n d\mu \longrightarrow 0$$

(b′)$\Longrightarrow$(a) From (b′) there exists constant $c>0$ such that for all $\phi \in L^1(\Gamma)$

$$\left| \int_\Gamma f(\gamma)\phi(\gamma) d\gamma \right| \leqq c \sup_{x \in X} |\hat{\phi}(x)| .$$

Then we define the bounded linear functional T on the subspace $A = \{\hat{\phi} \mid X ; \phi \in L^1(\Gamma)\}$ of $C_0(X)$ by

$$T(\hat{\phi}) = \int_\Gamma f\phi d\gamma \tag{1}$$

From the Hahn- Banach's theorem and the Riesz's representation theorem, there exists $\mu \in M(X)$ such that

$$T(\hat{\phi}) = \int_X \hat{\phi}(x)d\mu(x) \qquad (\phi \in L^1(\Gamma)). \tag{2}$$

From (1) and (2)

$$\int_\Gamma f\phi d\gamma = \int_X \int_\Gamma \phi(\gamma)(x, \gamma)d\gamma d\mu(x) = \int_\Gamma \hat{\mu}\phi d\gamma$$

for all $\phi \in L^1(\Gamma)$, so $f = \hat{\mu}$. q.e.d.

References

1) S. H. Friedberg, Functions which are Fourier-Stieltjes transforms, *Proc. Amer. Math. Soc.* *28* (1971), 451–452.
2) W. Rudin, Fourier Analysis on Groups, Interscience, 1962.