Optimal Accumulation of Capital and Experience

——Learning-by-Doing Revisited——

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Abstract

Reexamines the implications of learning-by-doing thesis by introducing a new measure of production experience. Through the analysis of a model of optimal accumulation of capital and experience, we find that some previous studies of learning-by-doing overemphasize the role of capital formatiom in the efficiency improvement; that optimal policy to guide our model economy requires more intricate preplanning to take account of the indirect control over the stock of experience. We also find that a common steady state growth rate in most learning-by-doing studies arises from the implicit public good property of the experience.

1. Introduction

Following the celebrated article by Arrow on the model of learning-by-doing[1], there have been many extensions and applications of the concept by Levhari, Shell, Sheshinski and others[2]. The purpose of this article is to reexamine the concept of learning-by-doing by introducing a new measure of production experience.

One of the main contributions of the learning-by-doing thesis is that the concept makes the production function as an intertemporally dependent process *with an endogenous dynamic increasing returns to scale.* The presence of such dynamic externality complicates the validity of a neoclassical market economy by the fact that the private rate of return on capital diverges from the corresponding social rate of return under the competitive environment. One of the implications is that a competitive economy tends to cause under-investment relative to a socially desirable level. The reason for this result is simply because private investors are *not fully* rewarded by their marginal product[3]. Although it remains true that they are not rewarded by their marginal product, whether they are entitled to be *fully* rewarded by their marginal product or not is a matter of discussion.

Since wage and rental incomes exhaust the total value of products at a time, they already share the fruit of efficiency generated by the past learning-by-doing activities which have been unconsciously conducted along the past production. In other words, a portion of national income essentially attributable to the stock of "production experience" that the economy has historically accumulated, belongs to the entire productive society as a gift from the preceding generations. Note that a competitive mechanism does not recognize the "production experience" as an independent factor of production, so that it exhaustively imputes the current total income, including the gift, to the current factor owners by their private rates of return. In this respect, there is not much reason for capital owners to appropriate more than their competitive rewards.

On the other hand, if an economy is to maximize the sum of *future* consumption stream, then the investment must be maintained at a level at which the marginal product of capital (the social rate of return on capital) is equal to the rate of interest (plus possibly a time discount rate). From this view point, a competitive solution indeed underinvests since the private rate of return on capital falls short of the social rate of return. If a competitive economy is to be guided to the optimal path under the latter view, a corrective fiscal policy subsidizing private investors so as to reward them by their marginal product is necessary. Therefore, such income distribution scheme is justifiable only by a social criterion which maximizes the *future* welfare of the economy.

It is evident, however, that the whole argument above, in particular the amount of subsidy to be assigned, depends upon the measurement of "production experience" for which cumulative gross investment has been conventionally adopted. I am going to present an alternative measure of the "production experience" which, I believe, is more reasonable. With the new measure, a corrective fiscal policy to guide a competitive economy requires more sophisticated program. The new measure also clearifies a confusion that new investment is as if the only source of efficiency improvement.

We will first consider a new measure of "production experience in section 2 and then construct a model of optimal accumulation of capital and experience in section 3. The characterization of optimal conditions is described in section 4. The optimal path are synthesized in section 5. Some concluding remarks are stated in section 6.

2. A Measure of Production Experience

Let us consider an aggregate neoclassical production function with a Harrod neutral, disembodied type technological progress.

$$Y=F(AL,\ K) \tag{1}$$

where Y=level of aggregate output, L=level of homogeneous labor, K=level of capital stock and A=technological efficiency per unit of labor. F is assumed to be essentially positive, quasi-concave, linear homogeneous and twice continuously differentiable in its arguments. Let G be a general measure of past production experience. Then the learning-by-doing thesis is described as

$$A=\tilde{A}(G)=G^{\eta} \tag{2}$$

where η=a constant elasticity of learning, $0<\eta<1$.

Arrow [1962] has taken the cumulative gross investment as a measure of production experience G

$$G=\int_{-\infty}^{t} \mathrm{I}(\tau)d\tau \tag{3}$$

where $\mathrm{I}(\tau)$=gross invesment at time $\tau \leqq t$.

However, what Arrow had originally in his mind seems to be the cumulative output of the past production

$$G=\int_{-\infty}^{t} Y(\tau)d\tau \tag{4}$$

Since Arrow assumed a fixed-coefficient type (vintage) production function, the substitution of gross investment for the gross output was a consistent alternative.

Levhari [1966] extended Arrow's analysis with a neoclassical production function (embodied type) not restricted to fixed coefficients, although the measure of production experience was approximated by the same cumulative gross investment (3). Sheshinski [1967] also adopted the measure (3) in his one sector model of optimal accumulation with a neoclassical aggregate production function (disembodied type). Within a disembodied technology, however, cumulative past investment such as (3) is not necessarily a good measure of production experience under factor substitution possibility unless the overall saving rate is constant overtime (by assumption in a descriptive model or in the steady state in an optimization model).

Sheshinski also introduces an alternative measure of production experience represented by overall capital-labor ratio, $k=K/L$, in his brief two-sector extension[4].

$$G=k=\frac{K}{L}=\frac{1}{L(t)}\int_{-\infty}^{t} I(\tau)d\tau \tag{5}$$

This representation greatly reduces the mathematical complexity but is less plausible as a measure of production experience. The intention of dividing capital by the current level of labor force seems, besides the above reason, to make level compatible with the technical efficiency term A which is defined for per unit of labor. If so, that is if the intention is to represent a per capita experience, then some portion of current labor force, especially recent entrants, may not share the older experiences which the elderly and retired workers had. Therefore, the representation of production experience per capita by (5) ignores the transition of generations along increasing labor force.

Coupled with a view that praises new investment as the main carrier of technological progress[5], measures of production experience using cumulated gross investment have been popular among most learning-by-doing models. However, since learning-by-doing is meant improvements in technique become available from the generation of experience within the production process itself, not only an expansion of production facility but also an intensity of production would affect the generation of experience. Then unless an approach is one of fixed-coefficient vintage type models, a measure of experience such as (3) ignores the effect of factor substitution history upon the experience and thus may give an impression that new investments are the only source of efficiency improvements under learning-by-doing technology.

The consideration leads us to modify the original measure of production experience represented by cumulated past output, (4). Taking account that the experience is shared by only those who worked at the time of production, it would be suitable to define,

$$G=\bar{y}=\rho e^{-\rho t}\int_{-\infty}^{t} e^{\rho\tau}[Y(\tau)/L(\tau)]\,d\tau \tag{6}$$

which can be interpreted as the stock of per capita production experience[6]. In fact, $\bar{y}$ is a weighted mean of past production experience per capita (average product of labor), $y=Y/L$, with the weights exponentially declining into the past. The backward discount factor $\rho>0$ represents a decay of production experience along the time passage mainly due to the historical transition of mortal labor forces[7]. The larger is ρ, the less weight is given to past per capita production in determining $\bar{y}(t)$, and vice versa. This new measure not only overcomes the above mentioned difficulties of using cumulated gross investment but also preserves the stressed link between investment and technical progress without overemphasizing the role. It is most important that this measure takes account of factor substitution possibilities under neoclassical production function. Note that these advantages can not be obtained without a sacrifice, that is, we have the stock variable "experience" in addition to the capital stock.

Differenciating (6) with respect to t (by Reibnitz rule[8]), we obtain

$$\dot{G}=\dot{\bar{y}}=\rho\left[\frac{Y}{L}-G\right]=\rho(y-\bar{y}) \tag{7}$$

where the dot over a variable means time derivative of that variable. The relation (7) indicates that the new measure of production experience per capita changes proportionally to the difference between the current output per capita and the measure itself. Since the latter is the weighted mean of past average products of labor, the measure admits a possibility that the production experience per capita may decline if there is a large reduction in current output per capita or if a period of declining output per capita prolongs.

The new measure is scaled in per capita basis so that it is compatible with the labor augmenting efficiency measure A. This implies that an absolute stock level of production experience could be thined by a fast increase in the labor force. However, unlike the controversial measure (5) which requires immediate decline in the efficiency of labor when the capital-labor ratio k declines, our measure (6) allows a small and short period of decline in current per capita output without requiring a decline in efficiency when the recent trend of output per capita is increasing. The moderate time lag between the new experience and the change in the direction of the stock of experience is due to the exponential weights attached to past experiences.

Let us define $g(\cdot)=F[\cdot,\ 1]=$average product of capital[9]. Then, $y=kg(\bar{y}^{\gamma}/k)=\tilde{y}(k,\ \bar{y})$ so that the average product of labor y is equal to the average product of capital times the capital-labor ratio. Note that by partial differenciation, we find

$$\frac{\partial \tilde{y}}{\partial k}=g(x)-xg'(x)>0, \qquad \frac{\partial \tilde{y}}{\partial \bar{y}}=\frac{\eta x g'(x)}{(\bar{y}/k)}>0$$

$$\frac{\partial^2 \tilde{y}}{\partial k^2}=\frac{x^2 g''(x)}{k}<0, \qquad \frac{\partial^2 \tilde{y}}{\partial \bar{y}^2}=-\frac{\eta[(1-\eta)xg'(x)-\eta x^2 g''(x)]}{(\bar{y}^2/k)}<0$$

$$\frac{\partial^2 \tilde{y}}{\partial k \partial \bar{y}}=-\frac{\eta x^2 g''(x)}{\bar{y}}>0 \tag{8}$$

where $x=\bar{y}^{\eta}/k$. Note also that

$$\left(\frac{\partial^2 \tilde{y}}{\partial k^2}\right)\left(\frac{\partial^2 \tilde{y}}{\partial \bar{y}^2}\right)-\left(\frac{\partial^2 \tilde{y}}{\partial k \partial \bar{y}}\right)^2=-\frac{\eta(1-\eta)x^3 g'(x)g''(x)}{\bar{y}^2}>0 \tag{9}$$

Therefore, $y=\tilde{y}(k,\ \bar{y})$ is concave in k and $\bar{y}$.

The dynamics of production experience (7) can be rewritten as

$$\dot{\bar{y}}=\rho[kg(\bar{y}^{\eta}/k)-\bar{y}] \tag{10}$$

In view of (8), we see that an increase in capital-labor ratio k enhances the production experience with a diminishing rate. Thus the link between investment and technological progress is preserved under factor substitution possibility.

It is interesting to see that per capita production function (or average productivity function) $\tilde{y}$ is nothing but a scaled redefinition of the production function on the $k-\bar{y}$ space.

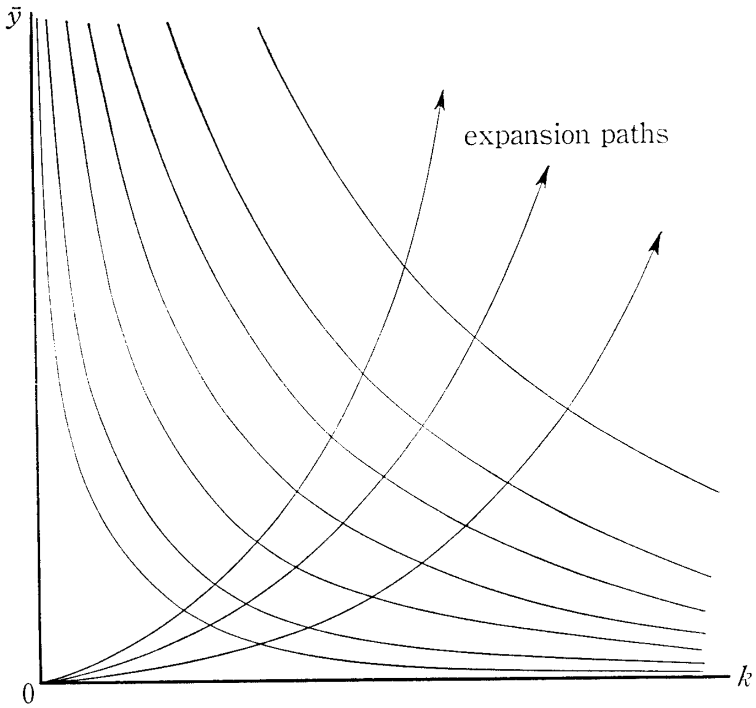

Figure 1 Per capita isoquants

$\tilde{y}(k,\ \bar{y})=kg(\bar{y}^{\gamma}/k)=F(\bar{y}^{\gamma},\ k)$

This representation enable us to draw per capita output isoquants on the state variable space of k and $\bar{y}$[10]. Since the function $\tilde{y}(k,\ \bar{y})$ is concave in k and $\bar{y}$, the isoquants are convex toward the origin. When superimposed on the phase diagrams analyzed later, the above property will enable us to identify whether a path is having increasing or decreasing per capita output (average productivity of labor) depending on the path is moving away from or toward the origin. Note that an expansion path with equal (effective) factor intensity will not be a straight ray through the origin but rather convex upward-sloping. See Figure 1. Note also that these isoquants are time invariant despite the assumed technological progress.

3. The Model

Let us consider an aggregate one sector model of optimal accumulation with disembodied learning-by-doing technological progress where the production experience is measured by (6). It is assumed that a central planning board tries to maximize an integral of discounted future per capita consumption stream subject to the neoclassical production system with learning-by-doing technology. The total population and thus total labor force is assumed to be increasing overtime at a constant rate $n>0$. The capital stock is subject to an exponential depreciation with a rate $\lambda>0$. Under the requirement of full employment of factors, the only mean of the central planning board to control the economy is the allocation of the total output $Y(t)$ between the consumption $C(t)$ and the investment $I(t)$. Our model, then, can be formulated as follows;

$$\mathrm{Max}\int_0^{\infty}[C(t)/L(t)]e^{-\delta t}dt \tag{11}$$

$$\text{s.t.}\quad Y(t)=C(t)+I(t) \tag{12}$$

$$Y(t)=F[A(t)L(t),\ K(t)] \tag{13}$$

$$A(t)=\tilde{A}[G(t)]=G(t)^{\gamma} \tag{14}$$

$$\dot{G}(t)=\rho[Y(t)/L(t)-G(t)] \tag{15}$$

$$\dot{K}(t)=I(t)-\lambda K(t) \tag{16}$$

$$\dot{L}(t)=nL(t) \tag{17}$$

where the historical profiles of capital stock and of labor force are given[11], and $\delta>0$

is a social discount rate. We also assume $\eta > \rho$[12].

Let $s = I/Y$ be the investment ratio. Then the above problem is equivalent to

$$\underset{0 \leq s \leq 1}{\mathrm{Max}} \int_0^\infty (1-s) k g(\bar{y}^\eta / k) e^{-\delta t} dt \tag{18}$$

$$\text{s.t.} \quad \dot{k} = s k g(\bar{y}^\eta / k) - (\lambda + n) k \tag{19}$$

$$\dot{\bar{y}} = \rho [k g(\bar{y}^\eta / k) - \bar{y}] \tag{20}$$

Where $k(0) = k_0$ and $\bar{y}(0) = \bar{y}_0$ are given historically. Note that $\bar{y} = G$.

The above problem can be solved by applying wellknown Maximum Principle. To do so, we first introduce the current value Hamiltonian $\mathscr{H}$ associated with the problem.

$$\begin{aligned} \mathscr{H} &= (1-s) k g(\bar{y}^\eta / \mathrm{k}) + p k [s g(\bar{y}^\eta / k) - (\lambda + n)] + q \rho [k g(\bar{y}^\eta / k) - \bar{y}] \\ &= [(1-s) + s p + \rho q] \tilde{y}(k, \ \bar{y}) - (\lambda + n) p k - \rho q \bar{y} \end{aligned} \tag{21}$$

$\mathscr{H}$ can be interpreted as the current value net national product per capita. The Hamiltonian multipliers p and q are the shadow prices of capital and production experience, respectively.

According to the Maximum principle, the optimality conditions can be stated as follows.

(i) $\mathscr{H}$ is maximized with respect to $s \in [0, \ 1]$ for all $t \geqq 0$. That is,

$$s \begin{cases} = 0 \\ \in [0, \ 1] \quad \text{according as} \quad p \lesseqqgtr 1. \\ = 1 \end{cases} \tag{22}$$

(ii) There exist piecewise continuous functions $p(t)$ and $q(t)$ such that

$$\dot{p} = \delta p - \frac{\partial \mathscr{H}}{\partial k} = (\delta + \lambda + n) p - [(1-s) + s p + \rho q] (\partial \tilde{y} / \partial k) \tag{23}$$

$$\dot{q} = \delta q - \frac{\partial \mathscr{H}}{\partial \bar{y}} = (\delta + \rho) q - [(1-s) + s p + \rho q] (\partial \tilde{y} / \partial \bar{y}) \tag{24}$$

$$\lim_{t \to \infty} p e^{-\delta t} = \lim_{t \to \infty} q e^{-\delta t} = 0 \tag{25}$$

(iii) $k(t)$ and $\bar{y}(t)$ satisfy (19) and (20) with the initial conditions. The above optimality conditions are not only necessary but also sufficient due to the fact that $\tilde{y}(k, \ \bar{y})$ is concave in k and $\bar{y}$.

Let us first examine the golden age solution where the output and the capital grow at a constant rate. Such stationary state values can be obtained by solving the conditions $p^* = 1$ and $\dot{k}^* = \dot{\bar{y}}^* = \dot{p}^* = \dot{q}^* = 0$. That is;

$$[(1-s^*) + s^* p^* + \rho q^*] = 1 + \rho q^* \tag{26}$$

$$s^*g(x^*)=\lambda+n \tag{27}$$

$$k^*g(x^*)=\bar{y}^* \tag{28}$$

$$(\delta+\lambda+n)p^*=[(1-s^*)+s^*p^*+\rho q^*][g(x^*)-x^*g'(x^*)] \tag{29}$$

$$(\delta+\rho)q^*=[(1-s^*)+s^*p^*+\rho q^*][\eta x^*g'(x^*)/(\bar{y}^*/k^*)] \tag{30}$$

where

$$x^*=\bar{y}^{*\eta}/k^* \tag{31}$$

Substituting equations (26) and (28) into (29) and (30), and solving for q^*, we obtain two expressions of q^*

$$q^*=\frac{(\delta+\lambda+n)-[g(x^*)-x^*g'(x^*)]}{\rho\,[g(x^*)-x^*g'(x^*)]}=\frac{\eta x^*g'(x^*)}{(\delta+\rho)g(x^*)-\rho\eta x^*g'(x^*)} \tag{32}$$

from which we have

$$\delta+\lambda+n=\left[\frac{g(x^*)}{g(x^*)-\left(\frac{\rho\eta}{\delta+\rho}\right)x^*g'(x^*)}\right][g(x^*)-x^*g'(x^*)]$$

$$=\left[\frac{g(x^*)}{g(x^*)-(1-a)x^*g'(x^*)}\right][g(x^*)-x^*g'(x^*)] \tag{33}$$

where $a=[\delta+(1-\eta)\rho]/(\delta+\rho)\in(0,\ 1)$

Now, let us define

$$\psi(x)=\left[\frac{g(x)}{g(x)-(1-a)xg'(x)}\right][g(x)-xg'(x)] \tag{34}$$

Then, $\psi(0)=0$, $\psi(\infty)=\infty$ and

$$\psi'(x)=\frac{(1-a)g'(x)[g(x)-xg'(x)]^2-ag(x)^2g''(x)}{[g(x)-(1-a)xg'(x)]^2}>0 \tag{35}$$

Therefore there exists an unique $x^*>0$ such that $\psi(x^*)=\delta+\lambda+n$, that is, the con-

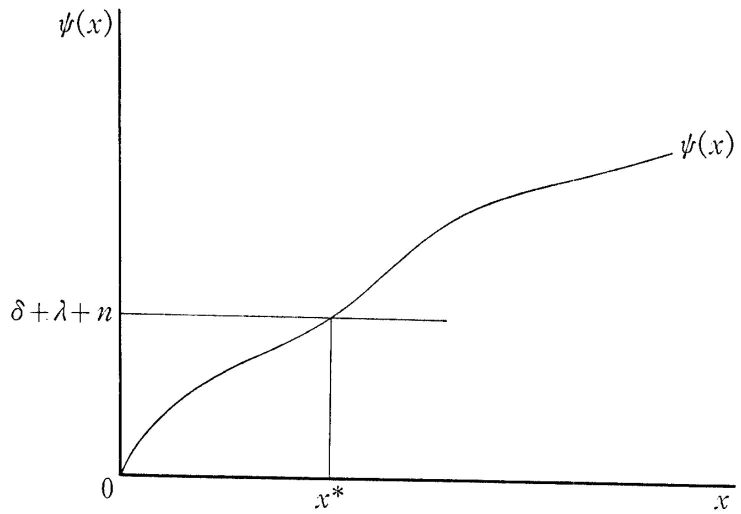

Figure 2 Determination of x^*

dition (33). See Figure 2 for the illustration. In fact, $\psi(x^*)$ is the marginal product of investment in the steady state.

The equation (32), then, uniquely determines q^*, and the condition (27) determines $s^*=(\lambda+n)/g(x^*)$. The remaining k^* and $\bar{y}^*$ can be obtained by solving the condition (28) with (31) yielding

$$k^*=\left[\frac{g(x^*)^\eta}{x^*}\right]^{\frac{1}{1-\eta}} \tag{36}$$

$$\bar{y}^*=\left[\frac{g(x^*)}{x^*}\right]^{\frac{1}{1-\eta}} \tag{37}$$

Geometrically, the long-run equiliblium state (k^*, $\bar{y}^*$) is given by the intersection of the curve $\bar{y}^\eta=x^*k$ and the ray $\bar{y}=g(x^*)k$ for a given x^* determined by (33). See Figure 3.

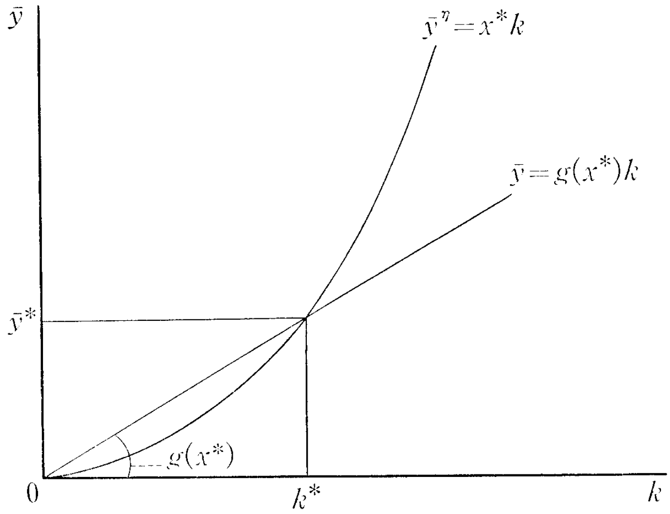

Figure 3 Determination of (k^*, $\bar{y}^*$)

To see the effects of parameter changes on the stationary state values, we first observe that an increase in η or in ρ decreases, while an increase in δ increases, the value of a. Therefore, by definition, the function $\psi(x)$ would shift upward if η or ρ is increased, or if δ is decreased, and vice versa. Then the equilibrium value of x^* would be smaller if η or ρ is increased, or if δ is decreased. Also an increase in λ, in n or in δ increases the value x^*. See Figure 4. Thus we conclude that

$$\frac{\partial x^*}{\partial \eta}<0,\ \frac{\partial x^*}{\partial \rho}<0,\ \frac{\partial x^*}{\partial \delta}>0,\ \frac{\partial x^*}{\partial \lambda}>0,\ \frac{\partial x^*}{\partial n}>0 \tag{38}$$

From the equations (36) and (37), we obtain, after calculation, that

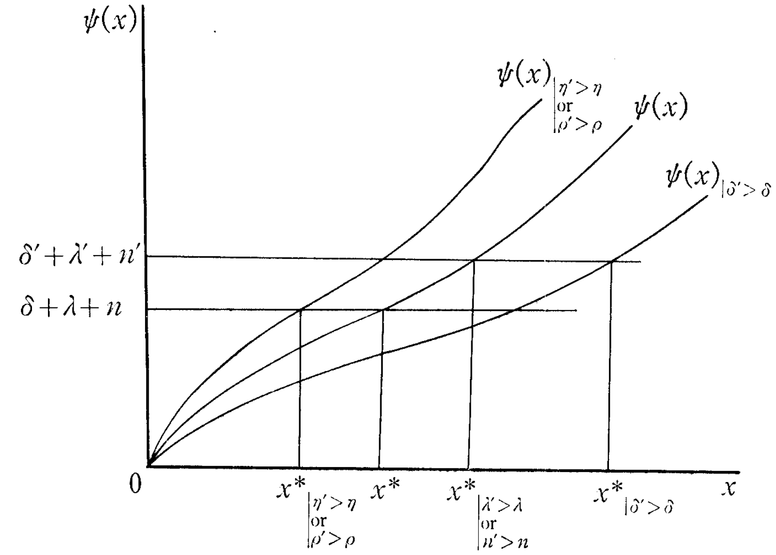

Figure 4 Effects of parameter changes on x^*

$$\frac{\partial k^*}{\partial \eta}=\frac{k^*}{(1-\eta)}\left\{\ln k+\ln g(x^*)-\left[\frac{g(x^*)-\eta x^* g'(x^*)}{x^* g(x^*)}\right]\left(\frac{\partial x^*}{\partial \eta}\right)\right\} \tag{39}$$

$$\frac{\partial k^*}{\partial \theta}=-\frac{k^*[g(x^*)-\eta x^* g'(x^*)]}{(1-\eta)x^* g(x^*)}\left(\frac{\partial x^*}{\partial \theta}\right) \quad \text{for } \theta=\rho,\ \delta,\ \lambda,\ n \tag{40}$$

$$\frac{\partial \bar{y}^*}{\partial \eta}=\frac{\bar{y}^*}{(1-\eta)}\left\{\ln \bar{y}^*-\left[\frac{g(x^*)-\eta x^* g'(x^*)}{x^* g(x^*)}\right]\left(\frac{\partial x^*}{\partial \eta}\right)\right\} \tag{41}$$

$$\frac{\partial \bar{y}^*}{\partial \theta}=-\frac{\bar{y}^*[g(x^*)-\eta x^* g'(x^*)]}{(1-\eta)x^* g(x^*)}\left(\frac{\partial x^*}{\partial \theta}\right) \quad \text{for } \theta=\rho,\ \delta,\ \lambda,\ n \tag{42}$$

Thus in view of (38), we have

$$\begin{gathered}\frac{\partial k^*}{\partial \eta}>0,\ \frac{\partial k^*}{\partial \rho}>0,\ \frac{\partial k^*}{\partial \delta}<0,\ \frac{\partial k^*}{\partial \lambda}<0,\ \frac{\partial k^*}{\partial n}<0\\ \frac{\partial \bar{y}^*}{\partial \eta}>0,\ \frac{\partial \bar{y}^*}{\partial \rho}>0,\ \frac{\partial \bar{y}^*}{\partial \delta}<0,\ \frac{\partial \bar{y}^*}{\partial \lambda}<0,\ \frac{\partial \bar{y}^*}{\partial n}<0\end{gathered} \tag{43}$$

These results imply that the faster the learning speed η, or the larger the exponential weight ρ on the past experience, the larger are the stationary state values of per capita caital k^* and the experience $\bar{y}^*$, and vice versa. Also the larger the values of δ (time discount rate), λ (capital depreciation rate) or n (labor growth rate), the smaller are the values of k^* and $\bar{y}^*$, and vice versa. It should be reminded that these results are steady state analysis. In particular, the larger value of ρ would mean the stock of experience consists of largely recent production experiences and thus less of older techniques involved. Note that in the steady state, $\bar{y}^*=y^*$.

Let us consider, for a while, a long-run equilibrium that the economy would attain if it were not centrally directed but under perfectly competitive markets with all the individuals having the utility integral (11). In a comparable balanced growth equilibrium where some but not all of income is saved, private investors would take investments so as to maintain,

$$\delta+\lambda+n=g(x^{**})-x^{**}g'(x^{**}) \tag{44}$$

where $g(x)-xg'(x)$ is the private rate of return on capital.

Comparing (44) with (33), we find that the effective labor intensity in the competitive solution x^{**} is larger than the socially optimal solution x^* since $\psi(x)>g(x)-xg'(x)$ for any $x>0$. See Figure 5. Since the steady-state solution for a competitive economy would also satisfy $\bar{y}^{\eta **}=x^{**}k^{**}$ and $\bar{y}^{**}=k^{**}g(x^{**})$, we have $k^{**}<k^*$ and $\bar{y}^{**}<\bar{y}^*$. Thus the competitive economy tends to settle to a steady-state in which the level of production $y^{**}=\tilde{y}(k^{**}, \bar{y}^{**})$ is below the socially optimal level $y^*=\tilde{y}(k^*, \bar{y}^*)$. It also implies that the competitive solution tends to under-invest since $s^{**}=(\lambda+n)/g(x^{**})<s^*=(\lambda+n)/g(x^*)$.

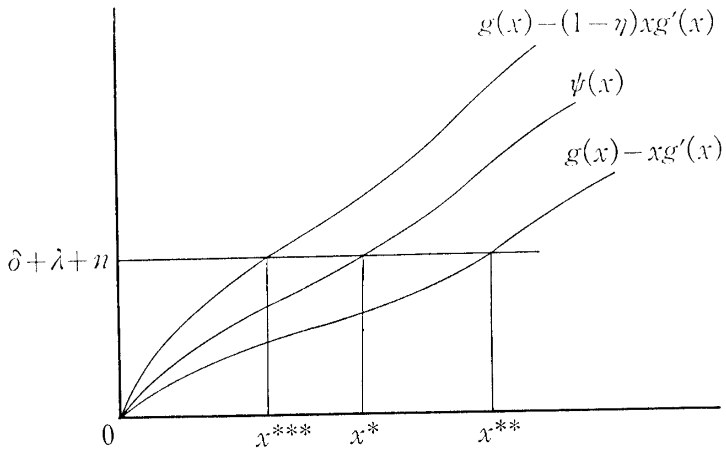

Figure 5 Relative sizes of x^*, x^{**} and x^{***}.

4. Characterization of Optimality Conditions

To analyze the time paths of the economy under the optimality conditions, it is necessary to divide the motion space into three phases depending on whether the value of p is less than, equal to or greater than unity. We will investigate the time paths of the economy in each of the phases and then synthesize the optimal paths over the phases.

Phase I ($p<1$)

When $p<1$, the imputed price of capital is less than that of consumption good, thus the maximization of per capita national income (the Hamiltonian) with respect to investment allocation requires $s=0$. In other words, the entire output is to be consumed and no investment is optimal. Then the dynamic system of the economy reduces to

$$\dot{k}=-(\lambda+n)k \tag{45}$$

$$\dot{\bar{y}}=\rho[kg(x)-\bar{y}] \tag{46}$$

$$\dot{p}=(\delta+\lambda+n)-(1+\rho q)[g(x)-xg'(x)] \tag{47}$$

$$\dot{q}=(\delta+\rho)q-(1+\rho q)[\eta xg'(x)/(\bar{y}/k)] \tag{48}$$

Since no investment takes place, the capital stock is left to decay as depreciation proceeds. The per capita capital k thus declines in this phase at the rate $\lambda+n$, the rate of depreciation plus the rate of population growth. As k declines, the per capita output $\tilde{y}(k, \bar{y})=kg(x)$ declines, which in turn reduces the learning-by-doing activity. Thus the rate of increase in the stock of production experience is declining in this phase. The stock level of experience, however, may not be declining until the per capita output $\tilde{y}(k, \bar{y})$ falls short of $\bar{y}$. That is, as far as the current average productivity of labor is higher than the past exponential mean, the stock of production experience per capita still grows at a declining rate. As time proceeds, the current average product of labor will decline to the past exponential mean. From that point, the stock of production experience starts to decline. Together with the prevailing decline in per capita capital, the output will be forced to decline further. The ultimate destination will be an economy without output and capital. Obviously, a path in this phase can not be optimal as a parmanent state. However, paths in this phase may constitute a part of the optimal path for a finite period of time when it is necessary to reduce the level of capital stock which is too large relative to the level of experience.

Formally, equation (45) describes the time path of k which is declining at a constant rate $\lambda+n$. The motion of $\bar{y}$ is described by (46). Notice that $\dot{\bar{y}}\gtreqless 0$ if and only if $kg(x)\gtreqless\bar{y}$. The locus of $\dot{\bar{y}}=0$ is given by

$$g(\bar{y}^{\eta}/k)=\bar{y}/k \tag{49}$$

Total differenciation of (49) yields

$$\frac{d\bar{y}}{dk}\Big|_{\dot{\bar{y}}=0} = \left[\frac{g(x)-xg'(x)}{g(x)-\eta xg'(x)}\right]\frac{\bar{y}}{k} > 0 \tag{50}$$

and

$$\frac{d^2\bar{y}}{dk^2}\Big|_{\dot{\bar{y}}=0} = -\frac{\bar{y}\{[g(x)-xg'(x)]^2-(1-\eta)xg(x)g''(x)\}}{k^2[g(x)-\eta xg'(x)]^2} \tag{51}$$

Thus the locus of $\dot{\bar{y}}=0$ in the $k-\bar{y}$ space in an increasing concave curve starting from the origin. Clearly, $\dot{\bar{y}}>0$ below the locus while $\dot{\bar{y}}<0$ above the locus. Together with the prevailing $\dot{k}<0$, time paths in this phase can be depicted as in Figure 6.

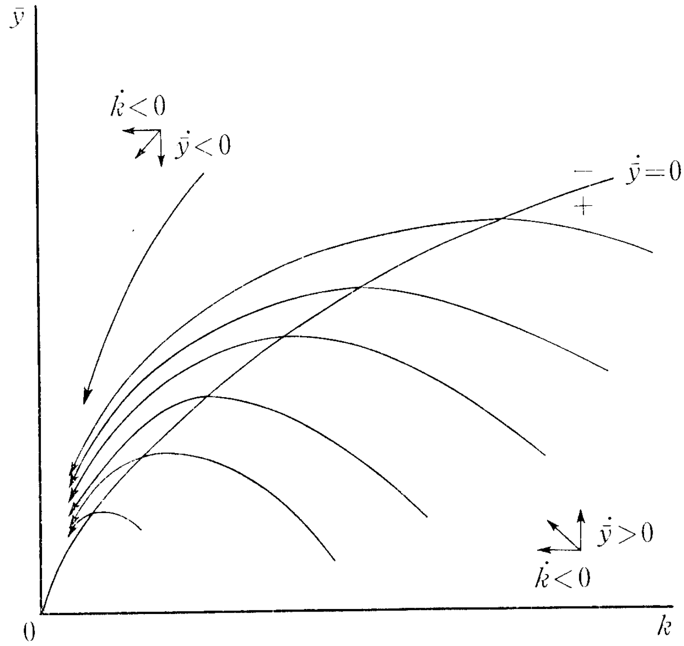

Figure 6 Movements of time paths in phase I

Notice that the dynamic behaviors of k and $\bar{y}$ are independent of equations (47) and (48). The behavior of corresponding imputed prices can be obtained by substituting the solution paths of k and $\bar{y}$ into (47) and (48). Below the $\dot{\bar{y}}=0$ locus, k is decreasing while $\bar{y}$ is increasing and thus $\partial\tilde{y}/\partial k = g(x)-xg'(x)$ is increasing while $\partial\tilde{y}/\partial\bar{y} = \eta xg'(x)/(\bar{y}/k)$ is decreasing by the concavity of $\tilde{y}(k, \bar{y})$. Therefore $\dot{p}$ is declining and $\dot{q}$ is increasing on a path below the $\dot{\bar{y}}=0$ locus. Upon crossing the $\dot{\bar{y}}=0$ locus, $\bar{y}$ starts declining and thus it becomes a counterforce against the increasing force on $g(x)-xg'(x)$ and decreasing force on $\eta xg'(x)/(\bar{y}/k)$ supported by declining k. The counterforce will first taper off the rate of decline in $\dot{p}$ and the rate of increase in $\dot{q}$ and then eventually takes over the force by k to lead $\dot{p}$ increasing and $\dot{q}$ decreasing as the path approaches the origin.

Also note that equation (49) is not affected by the phase division and thus remains the same in the other phases. In particular, the balanced growth path $(k^*, \bar{y}^*)$ must lie on the $\dot{\bar{y}}=0$ locus. This means that in the neighborhood of $(k^*, \bar{y}^*)$, phase I paths satisfy $\dot{p}_{\mathrm{I}}<0$, $\dot{k}_{\mathrm{I}}<0$, $\dot{q}_{\mathrm{I}}>0$ and $\dot{\bar{y}}_{\mathrm{I}}=0$[13].

Super-imposing the per capita output isoquants with the phase diagram obtained, we can easily identify that per capita output or average productivity of labor is declining on any path above the $\dot{\bar{y}}=0$ locus. On a path below the locus, on the other hand, it may increase despite the declining capital k. The latter phenomenon is due to the increase in the stock of production experience resulting from past production activities.

Phase II ($p>1$)

In phase II where $p>1$, the imputed price of capital is higher than that of consumption goods and thus the optimal policy requires to set $s=1$. Clearly, any phase II path can not be optimal for a long period of time since it means a path without consumption. However, the phase II paths may constitute a part of optimal paths for a finite period of time, and thus deserves to be studied. Under the condition $s=1$, the dynamic system of the economy can be described as;

$$\dot{k}=k[g(x)-(\lambda+n)] \tag{52}$$

$$\dot{\bar{y}}=\rho[kg(x)-\bar{y}] \tag{53}$$

$$\dot{p}=(\delta+\lambda+n)p-(p+\rho q)[g(x)-xg'(x)] \tag{54}$$

$$\dot{q}=(\delta+n)q-(p+\rho q)xg'(x)/(\bar{y}/k) \tag{55}$$

where $x=\bar{y}^{\eta}/k$.

Again, the time paths of k and $\bar{y}$, described by differential equations (52) and (53), respectively, are independent of p and q. Unlike phase I paths, however, the time paths of k and $\bar{y}$ are to be determined simultaneously by (52) and (53). Since $s=1$, the phase II paths require total output to be reinvested and thus all paths in this phase are on the course of full expansion, capacity wise. However, it is clear from (52) that if the average product of capital $g(x)$ falls short of $\lambda+n$, the rate of capital depreciaton plus the rate of population growth, then the capital per capita k may decline, and vice versa. Due to the monotone increasing property of $g(x)$, there exists an unique maximum sustainable level of effective labor intensity $x^{\max}$ such that

$$\lambda+n=g(x^{\max}) \tag{56}$$

Therefore, per capita capital k is increasing, decreasing or remains stationary according to whether the effective labor intensity at a state of economy $x=\bar{y}^{\gamma}/k$ is less than, greater than or equal to the maximum sustainable level $x^{\max}$. That is

$$\dot{k} \gtreqless 0 \text{ if and only if } x \lesseqgtr x^{\max} \tag{57}$$

The locus of $\dot{k}=0$ in $k-\bar{y}$ space is, then, given by

$$\bar{y}^{\gamma}=x^{\max} k \tag{58}$$

where $x^{\max}$ is a constant determined by (56).

Clearly, the locus starts at the origin, increasing with increasing slope. In fact. the locus is an expansion path across the isoquants of per capita output $\tilde{y}(k, \bar{y})$ with the constant effective labor intensity $x^{\max}$. In the region below this curve, where $x>x^{\max}$, the growth in labor force and the depreciation of capital exceed the full reproduction capacity of capital and thus the per capita capital k declines ($\dot{k}<0$). On the other hand, in the region above the curve, the converse is true and thus k increases ($\dot{k}>0$).

The behavioral property of the stock of production experience is the same as in phase I. The locus of $\dot{\bar{y}}=0$ intersects with the $\dot{k}=0$ curve only once in the interior of $k-\bar{y}$ quadrant. The intersection defines the stationary point ($k^{\max}$, $\bar{y}^{\max}$). Recalling that $\dot{\bar{y}}<0$ above and $\dot{\bar{y}}>0$ below the $\dot{\bar{y}}=0$ locus, the direction of the movement of phase II paths can now be identified.

In the region above both the $\dot{k}=0$ and $\dot{\bar{y}}=0$ curves, k is increasing while $\bar{y}$ is declining. In the region below the both curves, k is decreasing while $\bar{y}$ is increasing. In the region above the $\dot{k}=0$ locus but above the $\dot{\bar{y}}=0$ locus, both k and $\bar{y}$ are increasing toward the point ($k^{\max}$, $\bar{y}^{\max}$), while in the region below the $\dot{k}=0$ locus but above the $\dot{\bar{y}}=0$ locus, k and $\bar{y}$ are declining toward the same point. Therefore, the solution paths in this phase asymptotically converge to the point ($k^{\max}$, $\bar{y}^{\max}$). See Figure 7 for illustration.

Note that the balanced growth equilibrium values k^* and $\bar{y}^*$ are both smaller than the maximum sustainable values $k^{\max}$ and $\bar{y}^{\max}$, respectively. This can be verified by comparing (33) and (56). Since the point (k^*, $\bar{y}^*$) lies on the $\dot{\bar{y}}=0$ locus, we observe that $\dot{k}_{\mathrm{II}}>0$ and $\dot{\bar{y}}_{\mathrm{II}}=0$ in the neighborhood of that point.

As to the behavior of imputed prices, examination of (54) and (55) together with (8) shows that where k is increasing and $\bar{y}$ is decreasing, $\dot{p}$ is increasing and

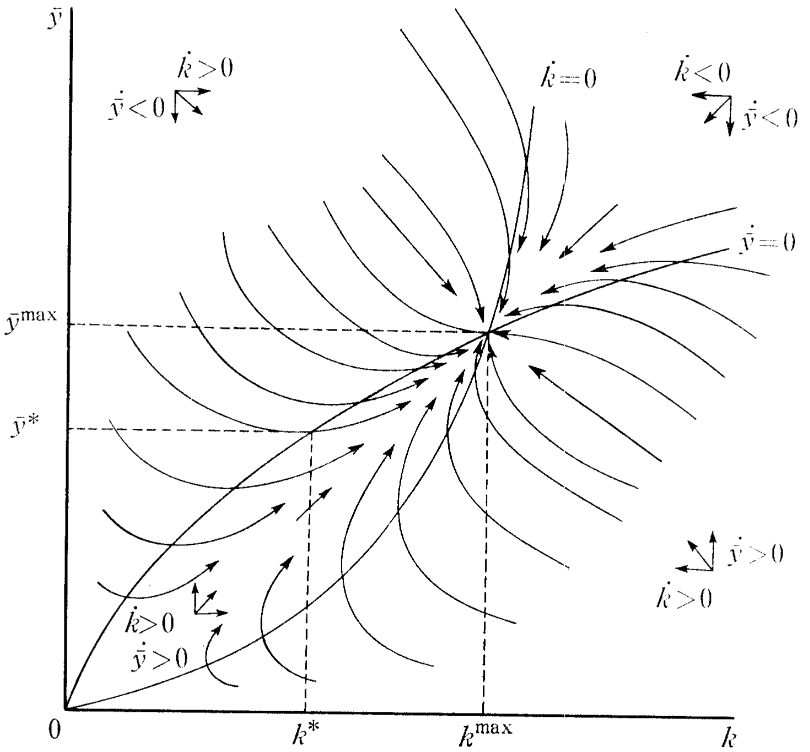

Figure 7 Movements of time paths in phase II

$\dot{q}$ is decreasing, and vice versa. Where k and $\bar{y}$ increase or decrease together, $\dot{p}$ and $\dot{q}$ may increase or decrease. However, in the neighborhood of $(k^*, \bar{y}^*)$, we see that $\dot{p}_{\mathrm{I}}>0$ and $\dot{q}_{\mathrm{I}}<0$ since $\dot{\bar{y}}_{\mathrm{I}}{}^*=0$.

Again, super-imposing the per capita output isoquants on the phase diagram, the behavior of average productivity of labor along any phase II path can be identified. Note that the behavior of average products of labor affects the dynamics of the stock of experience $\bar{y}$.

Phase III ($p=1$)

In phase III where $p=1$, the investment allocation rate s can be any value between 0 and 1 for the maximization of the Hamiltonian function $\mathscr{H}$. This is the case of singular extremum. The other optimality conditions reduce to

$$\dot{k}=k[sg(x)-(\lambda+n)] \tag{59}$$

$$\dot{\bar{y}}=\rho[kg(x)-\bar{y}] \tag{60}$$

$$\dot{p}=(\delta+\lambda+n)-(1+\rho q)[g(x)-xg'(x)] \tag{61}$$

$$\dot{q}=(\delta+\rho)q-(1+\rho q)\eta xg'(x)/(\bar{y}/k) \tag{62}$$

where $x=\bar{y}^{\eta}/k$ as before.

To determine the value of s, we use the fact that $\dot{p}$ must be equal to zero in order to sustain $p=1$. Therefore from equation (61), we obtain

$$q=\frac{(\delta+\lambda+n)-[g(x)-xg'(x)]}{\rho[g(x)-xg'(x)]}=\tilde{q}(x) \tag{63}$$

Differenciation of this equation with respect to t yields

$$\dot{q}=\frac{(\delta+\lambda+n)xg''(x)}{\rho[g(x)-xg'(x)]^2}\dot{x} \tag{64}$$

which is positive, equal to zero, or negative if and only if $\dot{x}\gtreqless 0$.

Next, since $\hat{x}=\eta\hat{y}-\hat{k}$, we obtain from (59) and (64) that

$$s=\frac{\lambda+n}{g(x)}+\frac{\eta\dot{y}}{\bar{y}g(x)}+\frac{\sigma}{\alpha g(x)}\,\frac{\rho}{1+\rho q}\dot{q}=\tilde{s}(k,\ \bar{y}) \tag{65}$$

where $\alpha=xg'(x)/g(x)\in[0,\ 1]$ is the share of marginal product of labor in the output and $\sigma=-g'(x)[g(x)-xg'(x)]/[xg(x)g''(x)]>0$ is the elasticity of substitution between the two factors. The relation (65) determines the value of s as a function of k and $\bar{y}$. We already know that if $\dot{y}=0$ and $\dot{q}=0$ (together with $\dot{p}=0$), then there exists an unique equilibrium value $x^*>0$ satisfying $\delta+\lambda+n=\psi(x^*)$, and the corresponding balanced growth equilibrium $(k^*,\ \bar{y}^*)$. Clearly, equation (65) satisfies $s=s^*=(\lambda+n)/g(x^*)$ under $\dot{y}=0$ and $\dot{q}=0$. Note that $\dot{y}=0$ and $\dot{q}=0$ imply $\dot{k}=0$. Substituting (65) back into (59), we may obtain the dynamic behavior of k. However, the locus of $\dot{k}=0$ is hard to illustrate without further assumption since it involves the elasticity of substitution σ. Instead, we look for the loci of $\dot{y}=0$ and $\dot{q}=0$. We know that the locus of $\dot{y}=0$ remains the same as in other phases. The locus of $\dot{q}=0$, on the other hand, is equivalent to the locus of $\dot{x}=0$ by (64), and thus it is nothing but a mapping of $\bar{y}^\eta=xk$ with constant x in the $k-\bar{y}$ space. The requirement that the locus of $\dot{x}=0$ (or $\dot{q}=0$) must intersect with the locus of $\dot{y}=0$ at the point $(k^*,\ \bar{y}^*)$ means, in fact, that the locus is given by $\bar{y}^\eta=x^*k$, See Figure 8.

More precisely, from (62) and (61), the condition $\dot{q}=0$ under $\dot{p}=0$ requires

$$q=\frac{\eta xg'(x)}{(\delta+\rho)(\bar{y}/k)-\rho\eta xg'(x)} \tag{66}$$

Equating with (63), we have (along $\dot{q}=0$)

$$\delta+\lambda+n=\left[\frac{(\bar{y}/k)}{(\bar{y}/k)-\left(\frac{\rho\eta}{\delta+\rho}\right)xg'(x)}\right][g(x)-xg'(x)]=\phi(k,\bar{y}) \tag{67}$$

The expression on the right hand side is similar to that of $\psi(x)$ in equation (33) except that $\bar{y}/k$ replaces $g(x)$. In fact,

$$\phi(k,\ \bar{y})\gtreqless\psi(x)\ \text{if and only if}\ \bar{y}/k\gtreqless g(x) \tag{68}$$

with $\phi(k^*, \bar{y}^*)=\phi(x^*)$. Notice that $\phi(k, \bar{y})$ can be written as

$$\phi(k, \bar{y})=\frac{g(x)-xg'(x)}{1-\left(\frac{\rho}{\delta+\rho}\right)\frac{\eta xg'(x)}{(\bar{y}/k)}}=\frac{(\partial\tilde{y}/\partial k)}{1-\left(\frac{\rho}{\delta+\rho}\right)(\partial\tilde{y}/\partial\bar{y})} \tag{69}$$

and is, in effect, the social rate of return on capital along constant shadow prices, $\dot{p}=\dot{q}=0$ (i.e., along an expansion path with constant x).

To find out behaviors of time paths in this phase, we first recall that $\dot{\bar{y}}<0$ above and $\dot{\bar{y}}>0$ below the $\dot{\bar{y}}=0$ locus. Thus the stock of per capita experience $\bar{y}$ is declining above the curve $\bar{y}/k=g(x)$ while it is increasing below the curve.

Next, observe that the effective labor intensity $\bar{y}^{\eta}/k$ is constant at x^* along $\dot{q}=0$. Then along the $\dot{q}=0$ locus, (65) reduces to

$$\begin{aligned} s_{|\dot{q}=0}&=\frac{\lambda+n}{g(x^*)}+\frac{\rho\eta}{g(x^*)}\left[\frac{kg(x^*)}{\bar{y}}-1\right]\\ &=\frac{\lambda+n}{g(x^*)}+\frac{\rho\eta}{g(x^*)}\left[\frac{g(x^*)}{x^{*\frac{1}{\eta}}\,k^{\frac{1-\eta}{\eta}}}-1\right] \end{aligned} \tag{70}$$

Since the first term on the right hand side is s^* and the second term vanishes at $(k^*, \bar{y}^*)$, we see that

$$k\gtreqless k^* \Rightarrow s_{|\dot{q}=0}\gtreqless s^* \Rightarrow \dot{k}_{|\dot{q}=0}\gtreqless 0 \tag{71}$$

Thus, per capita capital along the $\dot{q}=0$ locus is increasing to the right of k^* (or above the $\dot{\bar{y}}=0$ locus) and is decreasing to the left of k^* (or below the $\dot{\bar{y}}=0$ locus). Combined with the movement of $\bar{y}$, the motion *on* the $\dot{q}=0$ locus is toward the stationary state $(k^*, \bar{y}^*)$. See Figure 10.

Now differenciating (63) with respect to x, we have

$$\frac{dq}{dx}=\tilde{q}'(x)=\frac{(\delta+\lambda+n)xg''(x)}{\rho[g(x)-xg'(x)]^2}<0 \tag{72}$$

Thus $x\gtreqless x^*$ if and only if $q\lesseqgtr q^*$ where $q^*=\tilde{q}(x^*)$. This implies that at any point to the left of the $\dot{q}=0$ $(\dot{x}=0)$ locus, $x>x^*$ and thus $q<q^*$ while at any point to the right of the locus, $x<x^*$ and thus $q>q^*$.

Since $\tilde{q}(x)$ is a decreasing function of x, there exists an unique value $x^{**}>0$ such that $\tilde{q}(x^{**})=0$. That is, equation (44). The locus of $q=0$, then, is given by $\bar{y}^{\eta}/k=x^{**}$ which lies to the left of the $\dot{q}=0$ locus since $x^{**}>x^*$. x^{**} represents the competitive solution when all individuals are having the same utility function of the form (11). Under competitive mechanism, the market does not recognize the stock of production experience as a factor internal to the market but rather takes it as a granted environment. Therefore the market imputes zero price to the production experience $\bar{y}$. The locus of $q=0$ is then a competitive expansion path. In a steady state, the production experience will become also stationary so that the competitive economy will eventually settle to the point $(k^{**}, \bar{y}^{**})$ where the loci of $q=0$ and $\dot{\bar{y}}=0$ intersect. Clearly, $k^{**}<k^*$ and $\bar{y}^{**}<\bar{y}^*$. Note that along the $q=0$ locus,

$$\dot{q}\Big|_{q=0}=-\frac{\eta x^{**}g'(x^{**})}{\bar{y}/k}<0 \tag{73}$$

Thus the value of s at $(k^{**}, \bar{y}^{**})$, denoted by s^{**}, is less than the optimal value s^*. This can be confirmed easily from (65) as

$$s^{**}=\frac{\lambda+n}{g(x^{**})}-\frac{\rho\eta\sigma(x^{**})}{g(x^{**})}<\frac{\lambda+n}{g(x^*)}=s^* \tag{74}$$

Also note that to the right of the $q=0$ locus, we have $q>0$, while to the left of the locus, $q<0$. The latter case where the imputed price q is negative, is irrelevant to the phase III and need not be considered.

Now from equation (62), we have for $q>0$,

$$\hat{q}=(\delta+\rho)-\left(\frac{1+\rho q}{q}\right)\frac{\eta x g'(x)}{\bar{y}/k} \tag{75}$$

Differenciating this equation with respect to k and after manipulation, we obtain

$$\frac{\partial\hat{q}}{\partial k}=-\left(\frac{1+\rho q}{q}\right)^2\frac{\eta x^2 g''(x)}{\rho(\delta+\lambda+n)\bar{y}}[g(x)-(\delta+\lambda+n)]\lesseqgtr 0 \text{ as } g(x)\lesseqgtr\delta+\lambda+n \tag{76}$$

There exists an unique $x^+>0$ such that $g(x^+)=\delta+\lambda+n$. Therefore, $\partial\hat{q}/\partial k\lesseqgtr 0$ according to $x\lesseqgtr x^+$. Recall that $x^{\max}$ is defined by $g(x^{\max})=\delta+n$ and thus the locus of $x=x^+$, or equivalently the curve $\bar{y}^\eta=x^+k$, lies between the two loci of $x=x^{\max}$ and $\dot{q}=0$.

The relation (76) then indicates that at any point between the two loci $q=0$ and $\dot{q}=0$, we have $\hat{q}<0$, while at any point between the two loci $\dot{q}=0$ and $x=x^+$, we have $\hat{q}>0$. Recall also that $q<q^*$ to the left of $\dot{q}=0$ locus while $q>q^*$ to the right of the locus. Therefore, in the region bounded bythe two loci $q=0$ and $x=x^+$, the motion of time path *off* the $\dot{q}=0$ locus is away from the locus. This indicates that $(k^*, \bar{y}^*)$ is a saddle point with the $\dot{q}=0$ locus being the stable arms. There must exist, then, a pair of unstable branches of the saddle, one extending toward southwest but above the $\dot{\bar{y}}=0$ locus, and another extending toward northeast but below the $\dot{\bar{y}}=0$ locus. General motion around the equilibrium point $(k^*\ \bar{y}^*)$ is depicted in Figure 8.

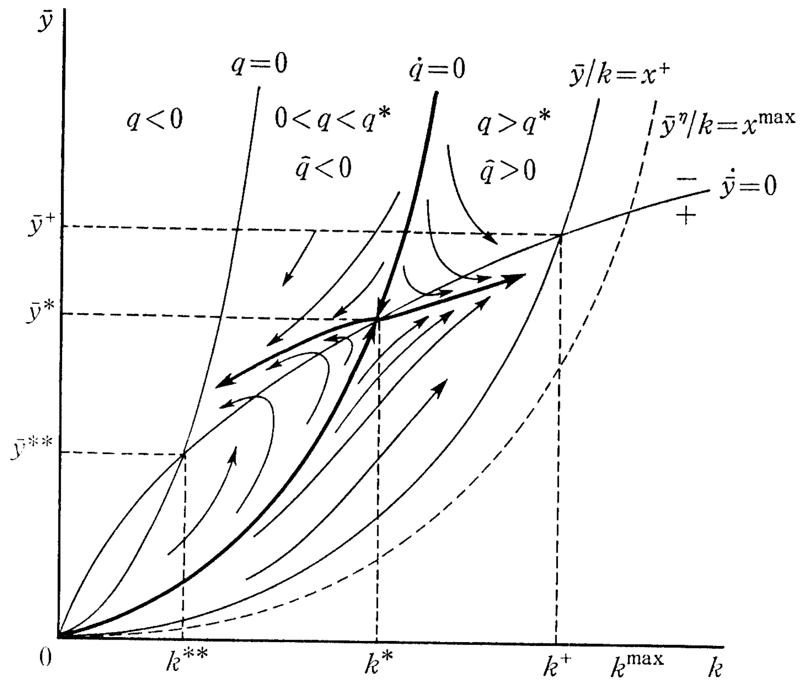

Figure 8 Movements of time paths in phase III

Note that the motion at points to the right of $x=x^+$ locus may not be monotone in k. The above analysis of phase III is sufficient to show that there are only two paths which asymptotically attain the optimal state $(k^*, \bar{y}^*)$. Therefore, the only eligible optimal paths in phase III are the ones following the $\dot{q}=0$ locus either from northeast or from southwest. Needlessly, the optimal paths in this phase maintain the optimal effective labor intensity x^*. The phase III path will be the final stage of any optimal trajectory.

5. Synthesis of Optimal Paths

Finally, we are ready to synthesize optimal paths for various initial states of the economy. First, if the initial state of the economy $(k_0, \bar{y}_0)$ happens to be at $(k^*, \bar{y}^*)$, then it is obviously optimal to remain there. The associated policy is to keep $s^*=(\lambda+n)/g(x^*)$ where $x^*=\bar{y}^{*\eta}/k^*=\bar{y}_0^\eta/k_0$. If the initial state (k_0, y_0) is on the $\dot{q}=0$ locus, that is if it happens that $\bar{y}_0^\eta/k_0=x^*$, then the optimal path for the economy is to follow the phase III optimal path toward $(k^*, \bar{y}^*)$. The corresponding optimal policy is to set $s=s_{|\dot{q}=0}$. The policy will lead the economy to the optimal steady state without any alteration in the effective labor intensity. Physical capital intensity would be monotonically increasing or decreasing depending on whether $k_0<k^*$ or $k_0>k^*$, respectively. The stock of production experience and thus the efficiency level also approach monotonically the steady state value in the same direction as k.

If the initial state of the economy $(k_0, \bar{y}_0)$ is somewhere off the $\dot{q}=0$ locus, then it is necessary to look for a backward solution in phase I or in phase II which switches into the $\dot{q}=0$ locus of phase III. Since the $\dot{q}=0$ locus of phase III is a convex power curve starting at the origin and lies left of the $\dot{k}=0$ locus ($\bar{y}^{\gamma}/k=x^{\max}$ curve) of phase II, it is always possible for any initial point to the left of the $\dot{q}=0$ locus to find a phase II path which pass through the initial point and reaches a point on the $\dot{q}=0$ locus. The part of phase II path will then become the first stage of the optimal trajectory for the initial point and when the path reaches the $\dot{q}=0$ locus, the optimal trajectory switches to the last stage of following the stable arm toward the steady state $(k^*, \bar{y}^*)$. The optimal policy along such trajectory is first to set $s=1$ until the economy attains the optimal effective labor intensity x^*. Then as soon as x^* is attained, it is optimal to set $s=s_{|\dot{q}=0}=(\lambda+n)/g(x^*)+\eta\rho[kg(x^*)/\bar{y}-1]/g(x^*)$ to follow the optimal phase III path toward $(k^*, \bar{y}^*)$. The first stage of such optimal path requires a rapid build up of capital stock, in fact without consumption. If the stock of experience is relatively low, the build up of capital stock would be still required in the last phase III stage with a slower rate. If the initial stock of experience is relatively large, then it may be optimal for per capita capital k to overshoot the target k^* in the first phase II path and then turns backward in the last phase III path. The optimal transition of the stock of experience, may or may not be monotone along the trajectory. However, if it is not monotone, it would involve a decline of the stock in the early stage of the trajectory.

If the initial state of the economy $(k_0, \bar{y}_0)$ lies somewhere to the right of the $\dot{q}=0$ locus, then the optimal path must take phase I path until it attains the optimal effective labor intensity x^*. Thereafter, the economy switches to phase III. The first stage of the optimal path requires no new investment $(s=0)$, allowing capital stock to depreciate. The stock of experience will be increasing or decreasing depending on whether the state of the economy is below or above the $\dot{\bar{y}}=0$ locus, respectively. The final phase III path requires s to be set to $s_{|\dot{q}=0}$ which leads the economy to the balanced growth state$(k^*, \bar{y}^*)$. It is also possible in this case that per capita capital k may start with a larger value than the optimal level k^* and decline to a value less than k^* and then approach k^* from below. The possibility of such non-

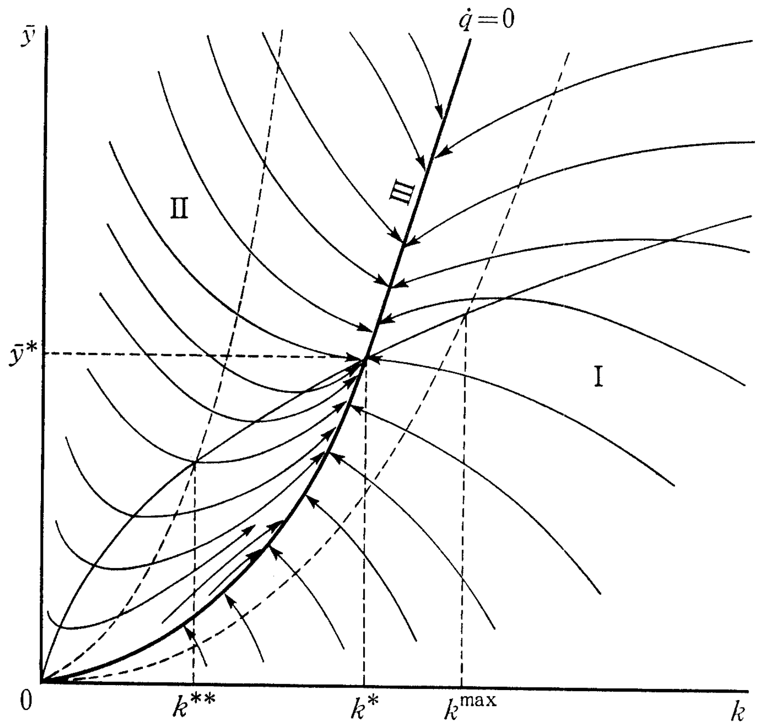

Figure 9 Synthesis of optimal paths

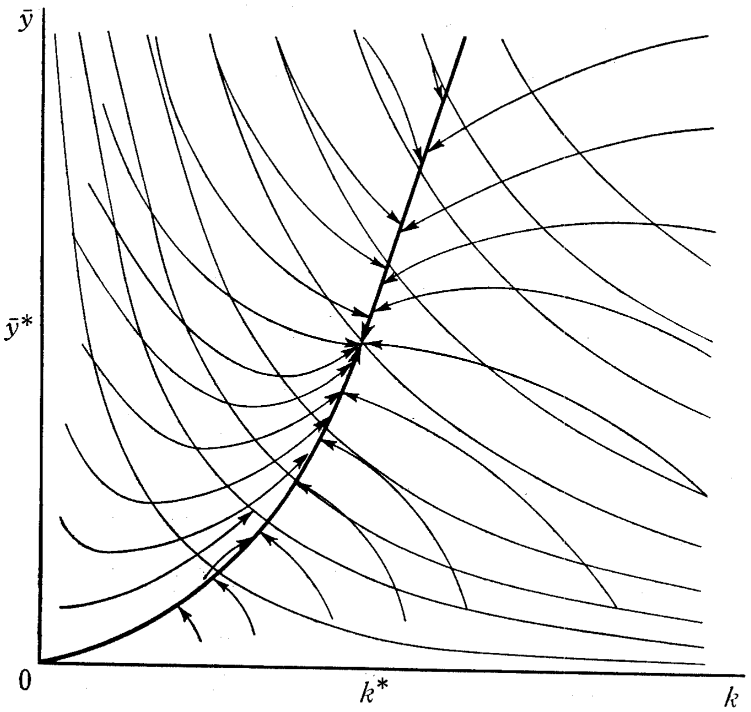

Figure 10 Optimal paths and per capita isoquants

monotone behavior in the capital-labor ratio was not available in previous learning-by-doing models[14]. The synthesis of optimal paths is illustrated in Figure 9.

If we superimpose per capita output isoquants on the diagram of synthesized optimal paths, then a planner can easily find the production levels as well as the factor input levels along the optimal path on which the economy must travel. See Figure 10.

6. Concluding Remarks

As we have seen, the adoption of an alternative measure of experience (6) requires more complex program to attain the socially optimal state than the one previously studied. It would be clear that the stock of production experience can be manipulated only indirectly through recent history of productive activities. This point is important for a planner of the central planning board who considers some corrective fiscal policy against the competitive market. Unlike previous models with the stock of experience measured by the past accumulation of gross investment, there does not exist, in our model, an instantaneously effective sudsidy scheme which transfers the state of economy right onto the optimal trajectory. In other words, the authority may have a direct control over the movements in physical capital-labor ratio k within a feasible range of investment allocation, but a desired movements in the stock of production exerience $\bar{y}$ can only be attained through a series of carefully calculated corrective investment allocations.

In practice, if the present economy does not have the optimal effecitive labor intensity x^*, then the intended investment subsidey scheme must result in either $s=1$ or $s=0$, that is, no consumption or no investment depending on whether the economy has $x>x^*$ or $x<x^*$, respectively. Although this result follows from the assumptionof a linearity of instantaneous utility function, the "bang-bang" solutions of this kind are the only controls found on the optimal paths under Sheshinski's version (with cumulated gross investment as a measure of experience)[15]. A practical tax-subsidy scheme works only when after the economy attained the phase III optimal path where $x=x^*$ (only at the stationary point in Sheshinski's model). The subsidey scheme along the final phase must still be calculated by taking account of the indirect future impacts

on $\bar{y}$ so as to follow the optimal path toward the balanced growth path (k^*, $\bar{y}^*$). The amount of subsidy on the phase III optimal path can be obtained by calculating the difference between the social rate of return on capital and the private rate of return. That is,

$$\phi(k,\bar{y})-[g(x^*)-x^*g'(x^*)]=\frac{(\delta+\rho)(\bar{y}/k)}{(\delta+\rho)(\bar{y}/k)-\eta x^*g'(x^*)}$$

$$=\frac{1}{1-\left(\frac{\rho}{\delta+\rho}\right)(\partial\dot{\bar{y}}/\partial\bar{y})} \tag{77}$$

This formula can be simply rewritten as ·

$$\delta+\lambda+n-[g(x^*)-x^*g'(x^*)]>0 \tag{78}$$

since it is required that $\phi(k,\bar{y})=\delta+\lambda+n$ along the $\dot{q}=0$ locus. Needlessly, the calculation of x^* requires solving $\psi(x^*)=\delta+\lambda+n$.

It should be noted that our new measure (6) incorporates three distinct features over the conventional measure (3).

(i) adoption of ***output***, instatead of gross investment, in measuring the momentary learning-by-doing activity (or momentary production experience).

(ii) ***scaling*** of past production experience ***by the size of labor force at the time of experience.***

(iii) adoption of ***exponentially weighted mean*** over the past experience instead of simple cumulation.

The first feature most distinctively affects the result of our analysis by separating the role of experience from the one of capital accumulation. The reasoning Arrow had used in not adopting the feature (i) become some what subtle in disembodied technology[16]. Since Arrow wanted a possibility of continual learning even in the steady state of production, he argues that a new investment good produced and put in use would be capable of changing the production environment so that it would serve as a stimulus to learning even in a constant level of production. However, in a steady state, the level of new investment itself remains constant at a replacement level. In particular, in the world of disembodied technology, a new but identical replacement machine does not seem to have much impact on productive environments. Note that either the choice is output or gross investment, the cumulative level as an index of experience would grow at a same rate in the steady state.

The second feature, on the other hand, is essentially responsible for our result that the stock of production experience (and thus efficiency level) remains constant in the steady state. Notice that this feature implicitly assumes non-transferable nature of experience without participating in production which seems to be an implication of learning-by-*doing*. In fact, if we construct an alternative measure of production experience without this feature, that is, assuming acquired experiences can be freely transferable to those who have not participated in the production, then the measure would be,

$$G=\rho e^{-\rho t}\int_{-\infty}^{t} e^{\rho\tau}Y(\tau)d\tau \tag{79}$$

As it turns out, with this measure, our main results remain mostly valid except for some minor modifications[17]. In particular, the stock of experience G and capital stock K as well as output Y in the steady state grow at a common rate $n/(1-\eta)$ which is the same as the results of Arrow, Sheshinski and others. In fact, what makes technological progress possible in a steady state is not the choice of variable (gross investment or output) for the measure of experience but rather the underlying assumption with respect to the ***publicness*** of experience. In other words, if experience is a pure public good so that anyone can appropriate others' experience, then the service flow of the public good (experience) is proportional to the *size* of the labor force. Therefore even if the stock of experience is stationary in the steady state, the service flow of the experience grows with the size of labor force.

The third feature takes account of the fact that past experience would eventually decay due to such factors as transition of mortal labor force, technologial obsolescence and so forth. No one would believe unit production experience in ancient time is as productive as that of recent days. Note that it is immaterial whether we use the weighted mean (average) or the weighted sum (integral) as the stock of experience since the former is a constant proportion (ρ) of the latter. Although this feature has least significant impacts on our results, it allows the possibility of declining productivity which have not been available in previous models. In the real world, there are evidences that such phenomenon occurs in some industries or countries.

Although the theory of learning by doing is becoming almost classic to economists, a closer investigation in the theory reveals to us that there is more to learn in the nature of technological advancement in production processes. Since the deletion of the feature (ii) does not alter the general characteristics of optimal paths, our results suggest that optimal accumulation program under learning-by-doing requires more sophisticated strategies than what has been studied.

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Notes

1. Arrow [1962].
2. Levhari [1966], Shell [1966], Sheshinski [1967], Bardhan [1970], Ryder [1969].
3. Sheshinski [1967].
4. Sheshinski [1967].
5. See Shell [1966], for example. The view, however, is essentially on the matter of diffusion process of a given technology which may be exogeneous or endogenous. See

Hu [1969] for a model of such diffusion process under exogeneously improving technology.

6. Similar concept in per capita consumption experience was pioneered by Ryder and Heal [1973] for their intertemporally dependent utility function.
7. ρ may include such factors as obsolescence by new techniques, loss of productive knowledge by fogetting on fading, loss of untransferable knowledge and skills upon retirement.
8. Differenciation of integral function with respect to a parameter. See any advanced calculus text.
9. $g(\cdot)$satisfies, by the assumptions imposed on F, that $g'(\cdot)>0$, $g''(\cdot)<0$, $g(0)=g'(\infty)=0$ and $g(\infty)=g'(0)=\infty$.
10. Strictly speaking, $\bar{y}$ is an environmental factor and thus the definition of isoquant differs from the one of traditional analysis.
11. This implies all initial values $K(0)$, $L(0)$ and $G(0)$ are all known constants.
12. This means that the elasticity of learning out of experience is greater than the rate of decay of the experience.
13. The subscript I denotes the variable under phase I.
14. Under the measure (3) studied by Sheshinski [1967], the optimal trajectory for each initial state is wholly contained in the same phase it starts until it attains the steady state value k^* (our $1/x^*$), implying that the general optimal program for $k(0)\neq k^*$ is to set either $s=0$ or $s=1$. Note that there is only two phases in the model (which correspond to our phases I and II).
15. Arrow [1962] p. 157.
16. The difference lies in the assumption with regard to the publicness of experience. See Starrett [1977] for the relation between publicness and the returns to scale.
17. Brief outline of the analysis under the measure (79) is given in the appendix.

Appendix

The purpose of this appendix is to provide a brief description of a problem similar to the one in the text but with the alternative measure (79) of the production experience. The main difference between the two measures (79) and (6) is that in the former, the levels of past experience are undeflated by the size of labor force at the time of experience. This means that experience once acquired becomes pure public good thereafter and thus is appropriable by every production process. Our model now is identical to the problem (11) through (17) in the text except that equation (15) is replaced by

$$\dot{G}(t)=\rho[Y(t)-G(t)] \tag{80}$$

To save lengthy description, let us use the same intensive variable labels as in the text with different definitions. Let us define

$y=Y/(L^{\frac{1}{1-\eta}})$, $k=K/(L^{\frac{1}{1-\eta}})$, $\bar{y}=G/(L^{\frac{1}{1-\eta}})$ and $\tilde{y}(k,\bar{y})=F[\bar{y}^{\eta},k]=kg(x)$.

$s=I/Y$ and $x=AL/K=\bar{y}^{\eta}/k$ remains the same as in the text.

Under these intensive variables, our model reduces to

$$\underset{s\in[0,1]}{\mathrm{Max}}\int_0^{\infty}(1-s)\tilde{y}(k,\bar{y})e^{-[\delta-\theta]t}dt \tag{81}$$

$$\text{s.t.}\quad \dot{k}=s\tilde{y}(k,\bar{y})-\left(\frac{n}{1-\eta}+\lambda\right)k \tag{82}$$

$$\dot{\bar{y}}=\rho\tilde{y}(k,\bar{y})-\left(\frac{n}{1-\eta}+\rho\right)\bar{y} \tag{83}$$

where $k(0)=k_0$ and $\bar{y}(0)=\bar{y}_0$ are historically given and $\theta=\left(\frac{\eta}{1-\eta}\right)n$ is assumed to be less than δ.

The corresponding Hamiltonian function is given by

$$\mathscr{H}=[(1-s)+sp+\rho q]\tilde{y}(k,\bar{y})-\left(\frac{n}{1-\eta}+\lambda\right)pk-\left(\frac{n}{1-\eta}+\rho\right)q\bar{y} \tag{84}$$

Not that the implicit prices of capital and experience p and q, respectively, are not the same as the ones in the text though we use the same labels. The optimality conditions are given by

$$s\begin{cases}=0\\ \in[0,1] \quad \text{according as} \quad p\lesseqqgtr 1\\ =1\end{cases} \tag{85}$$

$$\dot{p}=(\delta+n+\lambda)p-[(1-s)+sp+\rho q](\partial\tilde{y}/\partial k),\quad \lim_{t\to\infty}p(t)e^{-[\delta-\theta]t}=0 \tag{86}$$

$$\dot{q}=(\delta+n+\rho)q-[(1-s)+sp+\rho q](\partial\tilde{y}/\partial\bar{y}),\quad \lim_{t\to\infty}q(t)e^{-[\delta-\theta]t}=0 \tag{87}$$

$$\dot{k}=s\tilde{y}(k,\bar{y})-\left(\frac{n}{1-\eta}-\lambda\right)k,\qquad k(0)=k_0 \tag{88}$$

$$\dot{\bar{y}}=\rho\tilde{y}(k,\bar{y})-\left(\frac{n}{1-\eta}+\rho\right)\bar{y},\qquad \bar{y}(0)=\bar{y}_0 \tag{89}$$

where $\tilde{y}(k,\bar{y})$ has the same properties described in (8) and (9) and thus it is concave in k and $\bar{y}$. $\tilde{y}(k,\bar{y})$ here is not a per capita production function but is a rescaled replica of the former by a factor of $L^{\frac{\eta}{1-\eta}}$. Because of the concavity of $\tilde{y}$ in k and $\bar{y}$, the above optimality conditions are also sufficient.

The golden age solution can be obtained by solving the conditions $p^*=1$, $\hat{p}^*=\hat{q}^*=0$ and $\hat{k}^*=\hat{\bar{y}}^*=0$. The solution yields

$$k^*=\left[\frac{[(1-\eta)\rho]^{\eta}g(x^*)^{\eta}}{[(1-\eta)\rho+n]^{\eta}x^*}\right]^{\frac{1}{1-\eta}}>0 \tag{90}$$

$$\bar{y}^*=\left[\frac{[(1-\eta)\rho]g(x^*)}{[(1-\eta)\rho+n]x^*}\right]^{\frac{1}{1-\eta}}>0 \tag{91}$$

$$s^*=\frac{(1-\eta)\lambda+n}{(1-\eta)g(x^*)}\in(0,1) \tag{92}$$

$$p^*=1 \tag{93}$$

$$q^*=\frac{ax^*g'(x^*)}{[g(x^*)-ax^*g'(x^*)]},\qquad a=\frac{\eta\rho+\theta}{\delta+\rho+n} \tag{94}$$

where x^* is uniquely determined by

$$\delta+\lambda+n=\frac{g(x^*)[g(x^*)-x^*g'(x^*)]}{g(x^*)-ax^*g'(x^*)}=\psi(x^*) \tag{95}$$

Note that the constancy of k ahd $\bar{y}$ in the steady state implies that the capital stock K and the stock of experience G as well as the output Y all grow at a common rate $n/(1-\eta)$. Thus, despite the fact that we have replaced the cumulated gross investment by the exponentially weighted mean of past output levels as a measure of experience, the steady state growth rate remains the same as the one of Sheshinski. However, the dynamic characteristics of optimal paths, off the steady state path, are quite different from the one of Sheshinski and are similar to the one in our text.

In phase I where $p<1$, we must have $s=0$ so that the behaviors of k and $\bar{y}$ are governed by

$$\dot{k}=-\left(\frac{n}{1-\eta}+\lambda\right)k<0 \tag{96}$$

$$\dot{\bar{y}}=\rho kg(x)=\left(\frac{n}{1-\eta}+\rho\right)\bar{y}\gtreqless 0 \text{ as } \frac{\bar{y}}{k}\lesseqgtr\left[\frac{(1-\eta)\rho}{(1-\eta)\rho+n}\right]g(x) \tag{97}$$

The general motion of phase I paths is almost the same as that of phase I paths in the text illustrated in Figure 6 except that $\dot{\bar{y}}=0$ locus is given by $\bar{y}/k-\left[\frac{(1-\eta)\rho}{(1-\eta)\rho+n}\right]g(x)$. Note that the variable labels of both axis must be read in terms of new intensive variables.

Similarly, phase II paths where $p>1$ are governed by

$$\dot{k}=k\left[g(x)-\left(\frac{n}{1-\eta}+\lambda\right)\right]\gtreqless 0 \quad\text{as}\quad x\lesseqgtr x^{\max} \tag{98}$$

$$\dot{\bar{y}}=\rho kg(x)=\left(\frac{n}{1-\eta}+\rho\right)\bar{y}\gtreqless 0 \text{ as } \frac{\bar{y}}{k}\lesseqgtr\left[\frac{(1-\eta)\rho}{(1-\eta)\rho+n}\right]g(x) \tag{99}$$

where $x^{\max}$ is uniquely determined by $g(x)\frac{n}{1-\eta}+\lambda$.

The motion of phase II paths looks alike Figure 7 in the text. The movements of time paths asymptotically converge to the point $(k^{\max}, \bar{y}^{\max})$ where the two loci $\dot{k}=0$ and $\dot{\bar{y}}=0$ intersect. Note that in phase II, $s=1$. Phase I and II paths are the corner solutions with respect to the control s, and that the behaviors of k and $\bar{y}$ are independent of p and q.

In phase III where $p=1$, the case is one of singular extremum. Using the fact that $\dot{p}=0$ is necessary to remain in the phase, the locus of $\dot{q}=0$ is found to be the locus of k and $\bar{y}$ such that the imputed social rate of return on investment is equal to the economy's interest rate plus the rate of discount. i.e.,

$$\delta+\lambda+n=\frac{(\bar{y}/k)[g(x)-xg'(x)]}{(\bar{y}/k)-\left(\frac{\eta\rho}{\delta+n+\rho}\right)xg'(x)}=\phi(k,\bar{y}) \tag{100}$$

By the fact that $\phi(k^*, \bar{y}^*)=\phi(x^*)$, the $\dot{q}=0$ locus is nothing but the curve $\bar{y}^{\eta}/k=x^*$. The $\dot{\bar{y}}=0$ locus remains the same as in phases I and II. With arguments similar to the phase III in the text, The time paths in this phase can be illustrated analogus to Figure 8. The phase motion is asymptotically unstable around the saddle point $(k^*, \bar{y}^*)$ with the $\dot{q}=0$ locus being the stable branches. The stable arms from northeast and southwest are the only eligible paths to be optimal in this phase. The optimal investment allocation s along the stable arms is given by

$$\begin{aligned} s\Big|_{\dot{q}=0} &= s^*+\frac{\eta\hat{\bar{y}}}{g(x^*)} \\ &= \frac{(1-\eta)\lambda+n}{(1-\eta)g(x^*)}+\frac{\eta}{g(x^*)}\left[\frac{kg(x^*)}{\bar{y}}-\frac{(1-\eta)\rho+n}{(1-\eta)}\right] \end{aligned} \tag{101}$$

which leads the economy toward $(k^*, \bar{y}^*)$ without altering $x=x^*$. Note that the values of x^{**} and x^+ defined by $\delta+\lambda+n=g(x^{**})-x^{**}g'(x^{**})=g(x^+)$ remain the same as in the text and that $x^{\max}<x^+<x^*<x^{**}$ still holdes under the assumption $\delta>\theta$.

The synthesized optimal paths over the three phases can be seen as in Figure 9 with appropriate redefinitions of variables in behind. Therefore the general behavioral characteristics of optimal paths remains the same as that of optimal paths in the text.

(1984. 3. 11)