Proc. Sch. Sci. TOKAI UNIV.
60 (2025) 1–6
DOI: 10.18995/24352640.60.1

A List of 2-Elementary Lattices Appearing in the Theory of Involutions on $K3$ Surfaces

by

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(Received on Aug. 19, 2024; accepted on Oct. 6, 2024)

Abstract

We give a list of even indefinite 2-elementary lattices admitting a primitive embedding in an even unimodular lattice of signature $(3, 19)$. These are the lattices that give the classification of non-symplectic involutions on $K3$ surfaces.

Keywords: 2-elementary lattice, $K3$ surface, automorphism

2020 Mathematics Subject Classification: Primary 11E20, 15A63; Secondary 14J28, 14J50

1 Introduction

A *lattice* L is a free abelian group of finite rank equipped with a non-degenerate symmetric bilinear form, which will be denoted by $\langle\ ,\ \rangle$. Lattice theory gives fundamental techniques for algebraic geometry. For example, Mordell-Weil lattices in elliptic fibrations, automorphisms on $K3$ surfaces and Enriques surfaces. In particular Nikulin [7] gave the classification of non-symplectic involutions on $K3$ surfaces by characterizing their fixed loci in terms of the invariants of special lattices. These special lattices are 2-elementary lattices. This means that the classification of non-symplectic involutions on $K3$ surfaces is reduced exactly the classification of even indefinite 2-elementary lattices.

Indeed even indefinite 2-elementary lattices were classified by [6]. Then there exists a triangular graph in [7, page 1434] in which the invariants of 2-elementary lattices appearing in the theory of non-symplectic involutions on $K3$ surfaces are plotted. But it does not describe concrete lattices.

In this paper, our purpose is to give Table 1 in Theorem 3.2, hence to concretely write down the 75 lattices that appear in the classification of non-symplectic involutions on $K3$ surfaces. It allows us to make geometric considerations, such as constructions of elliptic fibrations of $K3$ surfaces with non-symplectic involutions.

* This work was supported by JSPS KAKENHI Grant Number JP23K03036.

Remark 1.1 1. For a odd prime number p, there exit lists of p-elementary lattices appearing in the theory of non-symplectic automorphisms on $K3$ surfaces. See [1], [2], and [9].

2. In this paper, we do not treat details of $K3$ surfaces. See [3], [4], and [5].

2 The classification of 2-elementary lattices

In this section, we review the classification of even indefinite 2-elementary lattices due to Nikulin [6]. For details about foundations of lattices, see [3, 8], and so on.

Let $L = (L, \langle\ ,\ \rangle)$ be a lattice of rank r. The bilinear form $\langle\ ,\ \rangle$ determines a canonical embedding $L \subset L^* = \mathrm{Hom}(L, \mathbb{Z})$. We denote by A_L the factor group L^*/L which is a finite abelian group. A lattice L is called *2-elementary* if $A_L \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$, where a is the minimal number of generator of A_L. In the following, we treat only even indefinite lattice. Hence for any $x \in L$, the value of $\langle x, x\rangle$ is even and the signature of $\langle\ ,\ \rangle$ is indefinite.

We denote by A_m, D_n, E_l the even negative-definite root lattice of type A_m, D_n, E_l respectively, and by U the even indefinite unimodular lattice of rank 2. For a lattice L, $L(m)$ is the lattice whose bilinear form is the one on L multiplied by m.

Example 2.1 1. An even unimodular lattice is 2-elementary with $a = 0$.

2. Lattices $E_8(2)$ and $U(2)$ are 2-elementary with $a = 8$ and $a = 2$ respectively.
3. Lattices A_1, D_n (n:even) and E_7 are 2-elementary of $a = 1, 2$ with $a = 1$ respectively.

Definition 2.2 Let $(L, \langle\ ,\ \rangle)$ be an even 2-elementary lattice and $(L^*, \langle\ ,\ \rangle_{L^*})$ its dual lattice. These induce the discriminant form $q : A_L \to \mathbb{Q}/2\mathbb{Z}$, $q(x + L) = \langle x, x\rangle_{L^*} + 2\mathbb{Z}$. Then we put

$$\delta_L = \begin{cases} 0 & \text{if } q(x+L) = 0, \forall x \in L^*, \\ 1 & \text{otherwise.} \end{cases}$$

Example 2.3 1. For an even unimodular lattice L, $L(m)$ has $\delta_{L(m)} = 0$.

2. $\delta_{A_1} = \delta_{E_7} = 1$.
3. It n is even then $\delta_{D_n} = 0$ holds.

Even indefinite 2-elementary lattices are classified by the following.

Proposition 2.4 ([6, Theorem 3.6.2]) An even indefinite 2-elementary lattice L is determined by the invariants (δ_L, t_+, t_-, a) where the pair (t_+, t_-) is the signature of L. An even 2-elementary lattice L with invariants (δ_L, t_+, t_-, a) exists if and only if all the following conditions are satisfied (it being assumed that $\delta_L = 0$ or 1, and that $t_+, t_-, a \geq 0$):

(1) $a \leq t_+ + t_-$;

(2) $t_+ + t_- + a \equiv 0 \mod 2$;

(3) $t_+ - t_- \equiv 0 \mod 4$ if $\delta_L = 0$;

(4) $\delta_L = 0, t_+ - t_- \equiv 0 \mod 8$ if $a = 0$;

(5) $t_+ - t_- \equiv \pm 1 \mod 8$ if $a = 1$;

(6) $\delta_L = 0$ if $a = 2, t_+ - t_- \equiv 4 \mod 8$;

(7) $t_+ - t_- \equiv 0 \mod 8$ if $\delta_L = 0$ and $a = t_+ + t_-$.

3 2-elementary lattices and $K3$ surfaces

Let X be a $K3$ surface. It is known that its 2-nd cohomology $H^2(X,\mathbb{Z})$ is torsion-free. Moreover we consider the cup product on $H^2(X,\mathbb{Z})$:

$$\langle\ ,\ \rangle : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z}.$$

Then the pair $(H^2(X,\mathbb{Z}), \langle\ ,\ \rangle)$ has a structure of a lattice. By Wu's formula, the Poincaré duality and the Hirzebruch index theorem, we see that $(H^2(X,\mathbb{Z}), \langle\ ,\ \rangle)$ is an even unimodular lattice of rank 22 with signature $(3,19)$.

Let L_{K3} be an even unimodular lattice of signature $(3,19)$. It is known that L_{K3} is isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2}$ by the classification of even unimodular indefinite lattices ([8], Chapter 5, §2).

Lemma 3.1 Let L, L_1 and L_2 be even 2-elementary lattices.

1. If the rank of L is r and $A_L \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$ then the inequality $a \leq r$ holds.
2. The equality $\delta_{L_1} = \delta_{L_2} = 0$ holds if and only if the equality $\delta_{L_1 \oplus L_2} = 0$ holds.

Proof. 1. It follows from $\mid L^*/L \mid = 2^a$, $\mid L^*/2L^* \mid = 2^r$ and $2L^* \subset L \subset L^*$.

2. It follows from $(L_1 \oplus L_2)^*/(L_1 \oplus L_2) \simeq (L_1^* \oplus L_2^*)/(L_1 \oplus L_2) \simeq A_{L_1} \oplus A_{L_2}$.

$\square$

We have the following Theorem by Proposition 2.4, Lemma 3.1, Example 2.1 and Example 2.3.

Theorem 3.2 Let S be an even hyperbolic 2-elementary lattice admitting a primitive embedding in L_{K3}. Let T be the orthogonal complement of S in L_{K3}. Then the following tables give all even hyperbolic 2-elementary lattices admitting a primitive embedding in L_{K3} (see [6] Sec.1, part 12°).

rank $S = 1$			
a	δ	S	T
1	0	$A_1(-1)$	$U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_1$

rank $S = 2$			
a	δ	S	T
0	0	U	$U^{\oplus 2} \oplus E_8^{\oplus 2}$
2	0	$U(2)$	$U \oplus U(2) \oplus E_8^{\oplus 2}$
2	1	$A_1(-1) \oplus A_1$	$A_1(-1)^{\oplus 2} \oplus E_8^{\oplus 2}$

rank $S = 3$			
a	δ	S	T
1	1	$U \oplus A_1$	$U^{\oplus 2} \oplus E_8 \oplus E_7$
3	1	$U(2) \oplus A_1$	$U \oplus U(2) \oplus E_8 \oplus E_7$

rank $S = 4$			
a	δ	S	T
2	1	$U \oplus A_1^{\oplus 2}$	$U^{\oplus 2} \oplus E_7^{\oplus 2}$
4	1	$U(2) \oplus A_1^{\oplus 2}$	$U \oplus U(2) \oplus E_7^{\oplus 2}$

rank $S = 5$			
a	δ	S	T
3	1	$U \oplus A_1^{\oplus 3}$	$U \oplus A_1(-1) \oplus E_7^{\oplus 2}$
5	1	$U(2) \oplus A_1^{\oplus 3}$	$U(2) \oplus A_1(-1) \oplus E_7^{\oplus 2}$

rank $S = 6$			
a	δ	S	T
2	0	$U \oplus D_4$	$U^{\oplus 2} \oplus E_8 \oplus D_4$
4	0	$U(2) \oplus D_4$	$U(2)^{\oplus 2} \oplus E_8 \oplus D_4$
4	1	$U \oplus A_1^{\oplus 4}$	$A_1(-1)^{\oplus 2} \oplus E_7^{\oplus 2}$
6	1	$U(2) \oplus A_1^{\oplus 4}$	$U(2)^{\oplus 2} \oplus E_7^{\oplus 2}$

rank $S = 7$			
a	δ	S	T
3	1	$U \oplus D_4 \oplus A_1$	$U^{\oplus 2} \oplus E_7 \oplus D_4$
5	1	$U \oplus A_1^{\oplus 5}$	$U^{\oplus 2} \oplus E_8 \oplus A_1^{\oplus 5}$
7	1	$U(2) \oplus A_1^{\oplus 5}$	$U \oplus U(2) \oplus E_8 \oplus A_1^{\oplus 5}$

rank $S = 8$			
a	δ	S	T
2	1	$A_1(-1) \oplus E_7$	$U^{\oplus 2} \oplus E_8 \oplus A_1^{\oplus 2}$
4	1	$U \oplus D_4 \oplus A_1^{\oplus 2}$	$U \oplus U(2) \oplus E_8 \oplus A_1^{\oplus 2}$
6	1	$U \oplus A_1^{\oplus 6}$	$U(2)^{\oplus 2} \oplus E_8 \oplus A_1^{\oplus 2}$
8	1	$U(2) \oplus A_1^{\oplus 6}$	$U(2)^{\oplus 2} \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$

rank $S = 9$			
a	δ	S	T
1	1	$U \oplus E_7$	$U^{\oplus 2} \oplus E_8 \oplus A_1$
3	1	$U(2) \oplus E_7$	$U \oplus U(2) \oplus E_8 \oplus A_1$
5	1	$U(2) \oplus D_4 \oplus A_1^{\oplus 3}$	$U^{\oplus 2} \oplus D_4^{\oplus 2} \oplus A_1$
7	1	$U \oplus A_1^{\oplus 7}$	$U \oplus U(2) \oplus D_4^{\oplus 2} \oplus A_1$
9	1	$U(2) \oplus A_1^{\oplus 7}$	$U^{\oplus 2} \oplus E_8(2) \oplus A_1$

rank $S = 10$			
a	δ	S	T
0	0	$U \oplus E_8$	$U^{\oplus 2} \oplus E_8$
2	0	$U(2) \oplus E_8$	$U \oplus U(2) \oplus E_8$
2	1	$A_1(-1) \oplus A_1 \oplus E_8$	$U \oplus A_1(-1) \oplus A_1 \oplus E_8$
4	0	$U \oplus D_4^{\oplus 2}$	$U^{\oplus 2} \oplus D_4^{\oplus 2}$
4	1	$A_1(-1) \oplus E_7 \oplus A_1^{\oplus 2}$	$U \oplus U(2) \oplus E_7 \oplus A_1$
6	0	$U(2) \oplus D_4^{\oplus 2}$	$U \oplus U(2) \oplus D_4^{\oplus 2}$
6	1	$U \oplus D_4 \oplus A_1^{\oplus 4}$	$U(2)^{\oplus 2} \oplus E_7 \oplus A_1$
8	0	$U \oplus E_8(2)$	$U^{\oplus 2} \oplus E_8(2)$
8	1	$U \oplus A_1^{\oplus 8}$	$U^{\oplus 2} \oplus A_1^{\oplus 8}$
10	0	$U(2) \oplus E_8(2)$	$U \oplus U(2) \oplus E_8(2)$
10	1	$U(2) \oplus A_1^{\oplus 8}$	$U \oplus U(2) \oplus A_1^{\oplus 8}$

rank $S = 11$			
a	δ	S	T
1	1	$U \oplus E_8 \oplus A_1$	$U^{\oplus 2} \oplus E_7$
3	1	$U(2) \oplus E_8 \oplus A_1$	$U \oplus U(2) \oplus E_7$
5	1	$U \oplus D_4^{\oplus 2} \oplus A_1$	$U \oplus A_1(-1) \oplus D_4^{\oplus 2}$
7	1	$U(2) \oplus D_4^{\oplus 2} \oplus A_1$	$U(2) \oplus A_1(-1) \oplus D_4^{\oplus 2}$
9	1	$U \oplus E_8(2) \oplus A_1$	$U \oplus A_1(-1) \oplus E_8(2)$
11	1	$U(2) \oplus E_8(2) \oplus A_1$	$U(2) \oplus A_1(-1) \oplus E_8(2)$

rank $S = 12$			
a	δ	S	T
2	1	$U \oplus E_8 \oplus A_1^{\oplus 2}$	$U \oplus A_1(-1) \oplus E_7$
4	1	$U(2) \oplus E_8 \oplus A_1^{\oplus 2}$	$U(2) \oplus A_1(-1) \oplus E_7$
6	1	$U \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$	$A_1(-1)^{\oplus 2} \oplus D_4^{\oplus 2}$
8	1	$U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 2}$	$U(2)^{\oplus 2} \oplus D_4^{\oplus 2}$
10	1	$U \oplus E_8(2) \oplus A_1^{\oplus 2}$	$A_1(-1)^{\oplus 2} \oplus E_8(2)$

rank $S = 13$			
a	δ	S	T
3	1	$U \oplus E_8 \oplus A_1^{\oplus 3}$	$U^{\oplus 2} \oplus D_4 \oplus A_1$
5	1	$U(2) \oplus E_8 \oplus A_1^{\oplus 3}$	$U \oplus U(2) \oplus D_4 \oplus A_1$
7	1	$U \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 3}$	$A_1(-1)^{\oplus 2} \oplus D_4^{\oplus 2}$
9	1	$U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 3}$	$U(2)^{\oplus 2} \oplus D_4^{\oplus 2}$

rank $S = 14$			
a	δ	S	T
2	0	$U \oplus E_8 \oplus D_4$	$U^{\oplus 2} \oplus D_4$
4	0	$U(2) \oplus E_8 \oplus D_4$	$U \oplus U(2) \oplus D_4$
4	1	$U \oplus E_8 \oplus A_1^{\oplus 4}$	$U^{\oplus 2} \oplus A_1^{\oplus 4}$
6	0	$U \oplus D_4^{\oplus 3}$	$U(2)^{\oplus 2} \oplus D_4$
6	1	$U \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 4}$	$U \oplus U(2) \oplus A_1^{\oplus 4}$
8	1	$U(2) \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 4}$	$U(2)^{\oplus 2} \oplus A_1^{\oplus 4}$

rank $S = 15$			
a	δ	S	T
3	1	$U \oplus E_8 \oplus D_4 \oplus A_1$	$U^{\oplus 2} \oplus A_1^{\oplus 3}$
5	1	$U \oplus E_8 \oplus A_1^{\oplus 5}$	$U \oplus U(2) \oplus A_1^{\oplus 3}$
7	1	$U \oplus D_4^{\oplus 3} \oplus A_1$	$U(2)^{\oplus 2} \oplus A_1^{\oplus 3}$

rank $S = 16$			
a	δ	S	T
2	1	$U \oplus E_7^{\oplus 2}$	$U^{\oplus 2} \oplus A_1^{\oplus 2}$
4	1	$U(2) \oplus E_7^{\oplus 2}$	$U \oplus U(2) \oplus A_1^{\oplus 2}$
6	1	$U \oplus E_8 \oplus A_1^{\oplus 6}$	$U(2)^{\oplus 2} \oplus A_1^{\oplus 2}$

rank $S = 17$			
a	δ	S	T
1	1	$U \oplus E_8 \oplus E_7$	$U^{\oplus 2} \oplus A_1$
3	1	$U(2) \oplus E_8 \oplus E_7$	$U \oplus U(2) \oplus A_1$
5	1	$U(2) \oplus E_7^{\oplus 2} \oplus A_1$	$U(2)^{\oplus 2} \oplus A_1$

rank $S = 18$			
a	δ	S	T
0	0	$U \oplus E_8^{\oplus 2}$	$U^{\oplus 2}$
2	0	$U(2) \oplus E_8^{\oplus 2}$	$U \oplus U(2)$
2	1	$A_1(-1) \oplus A_1 \oplus E_8^{\oplus 2}$	$U \oplus A_1(-1) \oplus A_1$
4	0	$U \oplus E_8 \oplus D_4^{\oplus 2}$	$U(2)^{\oplus 2}$
4	1	$U(2) \oplus E_8 \oplus E_7 \oplus A_1$	$A_1(-1)^{\oplus 2} \oplus A_1^{\oplus 2}$

rank $S = 19$			
a	δ	S	T
1	1	$U \oplus E_8^{\oplus 2} \oplus A_1$	$U \oplus A_1(-1)$
3	1	$U(2) \oplus E_8^{\oplus 2} \oplus A_1$	$U(2) \oplus A_1(-1)$

rank $S=20$			
a	δ	S	T
1	1	$U\oplus E_8^{\oplus 2}\oplus A_1^{\oplus 2}$	$A_1(-1)^{\oplus 2}$

Table 1: 2-elementary lattices

Remark 3.3 If X is a $K3$ surface with a non-symplectic involution ι then S is the invariant sublattice of $H^2(X,\mathbb{Z})$ with respect to the ι-action.

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