

A Note on the Nef Cones of Blow-ups at Points in a Product of Projective Spaces

by

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Abstract

We consider blow-ups of the product of a projective space and a projective line. We assume that the center consists of points in a special position, that is, all points lie on a fiber isomorphic to projective line. In this note, we describe explicitly the nef cones of such blow-ups.

Keywords: Projective space; Blow-up; Nef cone

1 Introduction

In algebraic geometry, several cones are defined on varieties. The study of their behavior under birational maps, such as blow-ups, is an interesting problem. For a projective variety X , one of the important cones is the nef cone of X , denoted by $\text{Nef}(X)$, whose elements are the numerical equivalence classes of divisors having non-negative intersection numbers with any curve on X .

The papers [1],[2] and [3] consider the blow-ups at *general* points in projective spaces or products of projective spaces, and describe several objects including the nef cones. In this note, we determine the nef cones of blow-ups at *special* points in the product of a projective space and a projective line.

We work over the field of complex numbers.

Theorem 1.1. *Let Y be the product $\mathbb{P}^n \times \mathbb{P}^1$ with $n \geq 1$. Let $\pi : X \rightarrow Y$ be the simultaneous blow-up at k points lying on a fiber of the projection $Y \rightarrow \mathbb{P}^n$. Let $E_1, \dots, E_k$ be the exceptional divisors. Then, we have*

$$\text{Nef}(X) = \mathbb{R}^+[H] + \mathbb{R}^+[L] + \sum_{r=1}^k \left(\sum_{1 \leq i_1 < \dots < i_r \leq k} \mathbb{R}^+[H + rL - E_{i_1} - \dots - E_{i_r}] \right),$$

where $H = \pi^* \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(1, 0)$ and $L = \pi^* \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(0, 1)$.

Remark. If $p : Y \rightarrow \mathbb{P}^n$ and $q : Y \rightarrow \mathbb{P}^1$ are two projections, we have $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(1, 0) = p^* \mathcal{O}_{\mathbb{P}^n}(1)$ and $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(0, 1) = q^* \mathcal{O}_{\mathbb{P}^1}(1)$.

2 Preliminaries

Let X be a smooth projective variety and let D be a divisor on X . The intersection number of D with a curve C on X can be defined as:

$$D \cdot C := \deg(D|_C).$$

Assume that D is a prime divisor. If C is not contained in D , we have $D \cdot C \geq 0$. If C is contained in D , the intersection number can be negative. Indeed, if X is the blow-up at a point of a smooth projective variety, we have $D \cdot C < 0$ for the exceptional divisor D and any curve $C \subset D$. We refer to [4] for the notions of divisors, blow-ups, etc.

Two divisors D_1 and D_2 on X are said to be *numerically equivalent* if $D_1 \cdot C = D_2 \cdot C$ for any curve C on X . The symbols $\sim$ and $\equiv$ stand for the linear and numerical equivalence, respectively. Note that $D_1 \sim D_2$ implies $D_1 \equiv D_2$. The numerical equivalence class of D is denoted by $[D]$.

From now on, we consider $\mathbb{R}$ -divisors, i.e. divisors whose coefficients are real numbers. The notion of intersection number is naturally extended to $\mathbb{R}$ -divisors. We define $a[A] + b[B] := [aA + bB]$ for real numbers a, b and $\mathbb{R}$ -divisors A, B . The set $N^1(X)_{\mathbb{R}}$ of all numerical equivalence classes of $\mathbb{R}$ -divisors on X forms a finite dimensional vector space over $\mathbb{R}$ and is called *Néron-Severi space*.

An $\mathbb{R}$ -divisor D on X is called *nef* if $D \cdot C \geq 0$ for any curve C on X . The *nef cone* of X is defined by

$$\text{Nef}(X) := \{[D] \in N^1(X)_{\mathbb{R}} \mid D \text{ is nef} \}.$$

In general, a subset $\mathcal{C}$ of a vector space V over $\mathbb{R}$ is a convex cone if $au + bv \in \mathcal{C}$ for any $u, v \in \mathcal{C}$ and $a, b \in \mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$. We see that $\text{Nef}(X)$ is a convex cone in the vector space $N^1(X)_{\mathbb{R}}$. Note also that the nef cone is closed in the Euclidean topology. See [5] for details on Néron-Severi spaces, nef cones, etc.

For example, we have $\text{Nef}(\mathbb{P}^n) = \mathbb{R}^+[H]$ where H is a hyperplane on $\mathbb{P}^n$. This follows from the fact that any divisor on $\mathbb{P}^n$ is linearly equivalent to a multiple of a hyperplane. In this note, the symbol $\mathcal{O}_{\mathbb{P}^n}(1)$ is used to denote a hyperplane on $\mathbb{P}^n$. Hence, we may write $\text{Nef}(\mathbb{P}^n) = \mathbb{R}^+[\mathcal{O}_{\mathbb{P}^n}(1)]$.

Another simple example is the following:

Proposition 2.1. *Let y be a point in $\mathbb{P}^n$ with $n \geq 2$. Let $\pi : X \rightarrow \mathbb{P}^n$ be the blow-up at y . Then, we have*

$$\text{Nef}(X) = \mathbb{R}^+[\pi^*\mathcal{O}_{\mathbb{P}^n}(1)] + \mathbb{R}^+[\pi^*\mathcal{O}_{\mathbb{P}^n}(1) - \text{Exc}(\pi)],$$

where $\text{Exc}(\pi) = \pi^{-1}(y)$ is the exceptional divisor.

Proof. Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is nef, so is the pull-back $H := \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Put $E := \text{Exc}(\pi)$. We show that $H - E$ is also nef. Let A be an irreducible curve on X . If $A \subset E$, we have $E \cdot A < 0$ and $H \cdot A = 0$. Hence $(H - E) \cdot A > 0$. Assume $A \not\subset E$. Consider the curve $B := \pi(A)$. Then, we have

$$(H - E) \cdot A = \mathcal{O}_{\mathbb{P}^n}(1) \cdot B - \text{mult}_y B \geq 0.$$

Conversely, let D be a nef divisor on X . Since $N^1(X)_{\mathbb{R}} = \mathbb{R}[H] + \mathbb{R}[E]$, there exist (not necessarily positive) real numbers x, y such that $D \equiv xH + yE$. Let m be a line in $\mathbb{P}^n$ passing through the point y . Let l be its strict transform (i.e. the closure of $\pi^{-1}(m \setminus \{y\})$ in X). Let e be a line in $E \simeq \mathbb{P}^{n-1}$. Then, we have

$$\begin{aligned} 0 \leq D \cdot l &= xH \cdot l + yE \cdot l = x + y, \\ 0 \leq D \cdot e &= xH \cdot e + yE \cdot e = -y. \end{aligned}$$

Hence, $[D] = (x + y)[H] + (-y)[H - E] \in \mathbb{R}^+[H] + \mathbb{R}^+[H - E]$. $\square$

The following is useful to determine the structure of nef cones.

Proposition 2.2. *Let X be a smooth projective variety and let $D_1, \dots, D_r$ be smooth prime divisors on X . Consider a divisor $D := m_1D_1 + \dots + m_rD_r$ with $m_1, \dots, m_r \geq 0$. Let N be a nef divisor on X . If $[(N + D)|_{D_i}] \in \text{Nef}(D_i)$ for any $i \in \{1, \dots, r\}$, then $[N + D] \in \text{Nef}(X)$.*

Proof. Let C be a curve on X . If $C \subset D_i$ for some $i \in \{1, \dots, r\}$, then

$$(N + D) \cdot C = (N + D)|_{D_i} \cdot C \geq 0.$$

If $C \not\subset D_i$ for any $i \in \{1, \dots, r\}$, then

$$(N + D) \cdot C = N \cdot C + m_1D_1 \cdot C + \dots + m_rD_r \cdot C \geq 0 + 0 + \dots + 0 = 0.$$

$\square$

3 Proof of Theorem

We use the following notation.

Put $Y := \mathbb{P}^n \times \mathbb{P}^1$ with $n \geq 1$. Let $p : Y \rightarrow \mathbb{P}^n$ and $q : Y \rightarrow \mathbb{P}^1$ be two natural projections. Note that $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(1, 0) = p^*\mathcal{O}_{\mathbb{P}^n}(1)$ and $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(0, 1) = q^*\mathcal{O}_{\mathbb{P}^1}(1)$ are nef divisors on Y . Let h_0 be a fiber of $p : Y \rightarrow \mathbb{P}^n$. Let $y_1, \dots, y_k$ be k distinct points on $h_0 \simeq \mathbb{P}^1$. Put $M_i := q^{-1}(q(y_i))$ and let m_i be a line in $M_i \simeq \mathbb{P}^n$ passing through the point y_i ($i = 1, \dots, k$).

Let $\pi : X \rightarrow Y$ be the blow-up at the points $y_1, \dots, y_k$. Put $E_i := \pi^{-1}(y_i)$ and let e_i be a line in $E_i \simeq \mathbb{P}^n$ ($i = 1, \dots, k$). Put $H := \pi^* \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(1, 0)$ and $L := \pi^* \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(0, 1)$. Note that H and L are nef divisors on X . Let h'_0 be the strict transform of h_0 . Let L_i and l_i respectively be the strict transforms of M_i and m_i ($i = 1, \dots, k$).

Lemma 3.1. *We have the following table of intersection numbers.*

	H	L	E_1	E_2	$\dots$	E_k
h'_0	0	1	1	1	$\dots$	1
l_1	1	0	1	0	$\dots$	0
l_2	1	0	0	1	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
l_k	1	0	0	0	$\dots$	1
e_1	0	0	-1	0	$\dots$	0
e_2	0	0	0	-1	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
e_k	0	0	0	0	$\dots$	-1

Proof. Since h_0 passes through $y_1, \dots, y_k$, we have $E_i \cdot h'_0 = 1$ for any $i \in \{1, \dots, k\}$. For each $i \in \{1, \dots, k\}$, we have $E_i \cdot l_i = 1$ and $E_j \cdot l_i = 0$ for $j \neq i$ because $y_i \in m_i$ and $y_j \notin m_i$. The other intersection numbers are obvious. $\square$

Lemma 3.2. *Let r be an integer such that $1 \leq r \leq k$. Assume $1 \leq i_1 < \dots < i_r \leq k$. Then, $H + rL - E_{i_1} - \dots - E_{i_r}$ is nef.*

Proof. Fix $i \in \{i_1, \dots, i_r\}$. If $n = 1$, we have the isomorphism $\pi|_{L_i} : L_i = l_i \rightarrow M_i = m_i \simeq \mathbb{P}^1$ and $(H - E_i)|_{L_i} = 0$, which is nef. If $n \geq 2$, the restriction map $\pi|_{L_i} : L_i \rightarrow M_i \simeq \mathbb{P}^n$ is the blow-up at y_i . By Proposition 2.1, $(H - E_i)|_{L_i} = (\pi|_{L_i})^* \mathcal{O}_{\mathbb{P}^n}(1) - \text{Exc}(\pi|_{L_i})$ is nef. For $j \neq i$, we have $L_j|_{L_i} \sim 0$. On the other hand, $L_i|_{L_i} \sim -E_i|_{L_i}$ because $L_i \sim L - E_i$ and $L|_{L_i} \sim 0$. Hence,

$$[H + L_{i_1} + \dots + L_{i_r}]|_{L_i} = [H - E_i]|_{L_i} \in \text{Nef}(L_i).$$

By Proposition 2.2, we conclude $[H + L_{i_1} + \dots + L_{i_r}] \in \text{Nef}(X)$. Thus, we are done, because

$$H + rL - E_{i_1} - \dots - E_{i_r} = H + (L - E_{i_1}) + \dots + (L - E_{i_r}) \sim H + L_{i_1} + \dots + L_{i_r}.$$

$\square$

Now, we prove Theorem. We put

$$\mathbb{R}^+[A, B, \dots, C] := \mathbb{R}^+[A] + \mathbb{R}^+[B] + \dots + \mathbb{R}^+[C].$$

Recall that H and L are nef. By Lemma 3.2 above, we have the inclusion:

$$\mathbb{R}^+[H, L] + \sum_{r=1}^k V_r \subset \text{Nef}(X),$$

where

$$V_r := \sum_{1 \leq i_1 < \dots < i_r \leq k} \mathbb{R}^+[H + rL - E_{i_1} - \dots - E_{i_r}].$$

Let D be an $\mathbb{R}$ -divisor on X . Since $N^1(X)_{\mathbb{R}} = \mathbb{R}[H] + \mathbb{R}[L] + \sum_{i=1}^k \mathbb{R}[E_i]$, there exist real numbers $a, b, c_1, \dots, c_k$ such that

$$D \equiv aH + bL - c_1E_1 - \dots - c_kE_k.$$

Assume that D is nef. In particular, for the curves $h'_0, l_1, \dots, l_k, e_1, \dots, e_k$, we have

$$D \cdot h'_0 \geq 0, \quad D \cdot l_i \geq 0, \quad D \cdot e_i \geq 0 \quad (i = 1, \dots, k).$$

By Lemma 3.1,

$$b - c_1 - \dots - c_k \geq 0, \quad a - c_i \geq 0, \quad c_i \geq 0 \quad (i = 1, \dots, k).$$

Consider the case in which $c_1 \geq c_2 \geq \dots \geq c_k$. We have

$$\begin{aligned} D &\equiv aH + bL - c_1E_1 - c_2E_2 - \dots - c_kE_k \\ &= (a - c_1)H + (b - c_1 - c_2 - \dots - c_k)L \\ &\quad + (c_1 - c_2)(H + L - E_1) \\ &\quad + (c_2 - c_3)(H + 2L - E_1 - E_2) \\ &\quad \vdots \\ &\quad + (c_{k-1} - c_k)(H + (k-1)L - E_1 - E_2 - \dots - E_{k-1}) \\ &\quad + c_k(H + kL - E_1 - E_2 - \dots - E_k). \end{aligned}$$

It follows that

$$[D] \in \mathbb{R}^+[H, L, H + L - E_1, \dots, H + kL - E_1 - E_2 - \dots - E_k].$$

Similarly, if $c_{\sigma(1)} \geq c_{\sigma(2)} \geq \dots \geq c_{\sigma(k)}$ for a permutation σ , we have

$$[D] \in \mathbb{R}^+[H, L, H + L - E_{\sigma(1)}, \dots, H + kL - E_{\sigma(1)} - E_{\sigma(2)} - \dots - E_{\sigma(k)}].$$

Therefore, in any case we have

$$[D] \in \mathbb{R}^+[H, L] + \sum_{r=1}^k V_r.$$

This completes the proof.

Remark. In general, a half line R in a convex cone $\mathcal{C}$ is called *extremal ray*, if $u + v \in R$ implies $u \in R$ or $v \in R$ for any $u, v \in \mathcal{C}$. We see that every half line appeared on the right side of the equality in Theorem is an extremal ray of the cone $\text{Nef}(X)$. Hence, the number of extremal rays of $\text{Nef}(X)$ is equal to $2 + \sum_{r=1}^k \binom{k}{r} = 2^k + 1$.

Note that the nef cone of the simultaneous blow-up at k points on a line in $\mathbb{P}^n$ ($n \geq 2$) consists of $k + 1$ extremal rays. See [6] Lemma 2.2, whose proof is valid for $n \geq 3$.

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