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A Three Dimensional Model of the Real Special Linear Group of Degree Two and its Applications

by

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Abstract

In this paper, a three dimensional model of $SL(2,\mathbb{R})$ is proposed. A bijection from $SL(2,\mathbb{R})$ to $\mathbb{R}^3 - \{(0,0,z) : z \in \mathbb{R}\}$ enables us to see the orbit of the exponential map as a space curve in $\mathbb{R}^3$. As applications, we can visualize the image of the exponential map, non-commutativity of matrix multiplication, and Lie algebra as tangent vectors around the unit matrix.

Keywords: Exponential map; Three dimensional model; Lie algebra

1 Introduction

The special linear group $SL(2,\mathbb{R})$ is the group of all real 2×2 matrices with determinant 1 ([1] p.25):

$$SL(2,\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) \;\middle|\; ad - bc = 1 \right\}.$$

The Lie algebra of $SL(2,\mathbb{R})$, denoted $\mathfrak{sl}(2,\mathbb{R})$, is the algebra of all real, traceless 2×2 matrices ([1] p.57):

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ X = \begin{pmatrix} \alpha & \beta + \gamma \\ -\beta + \gamma & -\alpha \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma \in \mathbb{R} \right\}.$$

The exponential map ([1] p.58)

$$exp : \; X \in \mathfrak{sl}(2,\mathbb{R}) \;\rightarrow\; exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \in SL(2,\mathbb{R})$$

is not surjective in $SL(2,\mathbb{R})$ ([1] p.99). The image of the exponential map is the following ([4] p.58, p.169):

$$\{M \in SL(2,\mathbb{R}) \mid \mathrm{Tr}M > -2\} \cup \{-I_2\}, \tag{1.1}$$

where I_2 is the unit matrix

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

How can we imagine the image of the exponential map in $SL(2,\mathbb{R})$? This question is the starting point of this research. In this paper, we will try to visualize the image of the exponential map with a bijection from $SL(2,\mathbb{R})$ to $\mathbb{R}^3 - \{(0,0,z) : z \in \mathbb{R}\}$.

In this model as in Figure 1(left), the unit matrix I_2 (resp. $-I_2$) corresponds to the point $(1,0,0)$

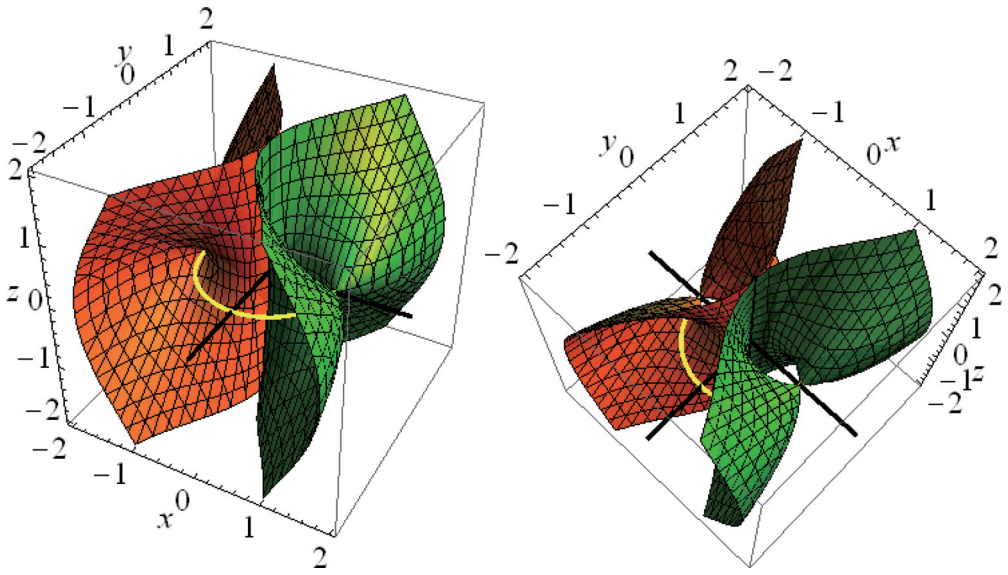

Figure 1: Images of the three dimensional model.

(resp. $(-1,0,0)$). The subgroup $SO(2)$ corresponds to the unit circle on the xy-plane. In Figure 1(left), there are two surfaces which look like tangent to each other in the z-axis. In fact, the left surface containing $(-1,0,0)$ is the boundary of the image of the exponential map. The right surface containing $(1,0,0)$ is the subset in which the value of the trace is equal to 2. In addition, as in Figure 1(right), we can see cone-like shape around $(1,0,0)$, and this shape has a deep relation with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

In Section 2, we will introduce a parametrization of $SL(2,\mathbb{R})$. With the parametrization, we will construct a bijection from $SL(2,\mathbb{R})$ to $\mathbb{R}^3 - \{(0,0,z) : z \in \mathbb{R}\}$. The three dimensional model has three axes, and each axis has geometric meaning; trace, asymmetry, and angle. We will see these geometric features in Section 3. In Section 4, we will investigate the orbit of the exponential map. A relation between this model and Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ will be cleared. Finally, we will see the non-commutativity of matrix multiplication with this model in Section 5. In this paper, we use two software "Mathematica 9" and "Cabri II plus" to make the figures.

2 A parameterization of $SL(2,\mathbb{R})$

The definition of $SL(2,\mathbb{R})$ is the following:

$$SL(2,\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) \;\middle|\; ad - bc = 1 \right\}.$$

A simple geometric image of $SL(2,\mathbb{R})$ is the set of parallelograms with area 1. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, we set $A = (a,b)$ and $B = (c,d)$. Then, the area of the parallelogram subtended by two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is equal to 1. We set $B' = (d,-c)$ be the rotated point of B about the origin O with angle $-\pi/2$. Set $T = ((a+d)/2, (b-c)/2)$ as the midpoint of A and B', then

$$OT^2 = \frac{(a+d)^2 + (b-c)^2}{4} = \frac{(a-d)^2 + (b+c)^2}{4} + 1 \geq 1,$$

hence, T is parametrized such as $T = (\cosh s \cos\varphi, \cosh s \sin\varphi)$ $(s \geq 0, \varphi \in [0, 2\pi))$. Let C be a circle centered at T with diameter AB'. The radius r_0 of circle C is

$$r_0 = AT = \sqrt{\frac{(a-d)^2 + (b+c)^2}{4}} = \sqrt{OT^2 - 1} = \sinh s$$

which means that C and the unit circle U centered at O intersect orthogonally because $OT^2 = AT^2 + 1$. Measuring the argument of $\overrightarrow{TA}$ from the x-axis with θ, we get a parameterization of $SL(2,\mathbb{R})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cosh s \cos\varphi + \sinh s \cos\theta & \cosh s \sin\varphi + \sinh s \sin\theta \\ -\cosh s \sin\varphi + \sinh s \sin\theta & \cosh s \cos\varphi - \sinh s \cos\theta \end{pmatrix},$$

($s \geq 0$, $\varphi \in [0, 2\pi)$, and $\theta \in [0, 2\pi)$). In this way, $SL(2,\mathbb{R})$ is parametrized with three parameters. With this parametrization, we will construct the three dimensional model in the next section.

3 Three dimensional model

The parametrization introduced in the previous section enables us to construct the three dimensional model. In the following discussion, we modify the parametrization such that θ is the argument from vector $\overrightarrow{OT}$ as in Figure 2:

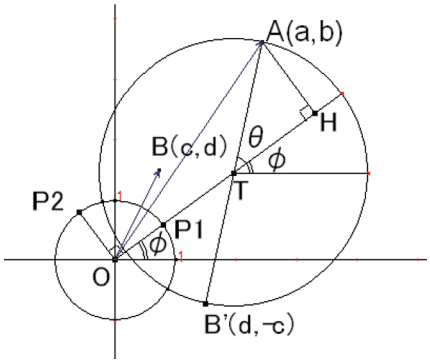

Figure 2: Modified parametrization.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cosh s \cos\varphi + \sinh s \cos(\varphi+\theta) & \cosh s \sin\varphi + \sinh s \sin(\varphi+\theta) \\ -\cosh s \sin\varphi + \sinh s \sin(\varphi+\theta) & \cosh s \cos\varphi - \sinh s \cos(\varphi+\theta) \end{pmatrix}. \tag{3.1}$$

The idea of the three dimensional model is a rotation of A around the vector $\overrightarrow{OT}$ with angle

$\pi/2$ in $\mathbb{R}^3$ by using the "Ampère's right hand screw rule". Under the modified parameters, $T = (\cosh s \cos\varphi, \cosh s \sin\varphi)$ and $r_0 = \sinh s$. Let $A_0 = (x, y, z)$ be the rotated point of A with angle $\pi/2$ around $\overrightarrow{OT}$ in $\mathbb{R}^3$. Let H be the foot of the perpendicular line through A perpendicular to line OT on the xy-plane as in Figure 2. Then,

$$\begin{aligned} OH &= OT + r_0 \cos\theta = \cosh s + \sinh s \cos\theta, \\ AH &= r_0 \sin\theta = \sinh s \sin\theta. \end{aligned}$$

Hence, the corresponding point $A_0 = (x, y, z)$ of A is given by the formula

$$(x, y, z) = ((\cosh s + \sinh s \cos\theta)\cos\varphi, (\cosh s + \sinh s \cos\theta)\sin\varphi, \sinh s \sin\theta)\,. \tag{3.2}$$

With this idea, we construct a map from $SL(2, \mathbb{R})$ to $\mathbb{R}^3 - \{(0, 0, z) : z \in \mathbb{R}\}$.

Theorem 3.1. *We define a map $f : SL(2, \mathbb{R}) \to \mathbb{R}^3 - \{(0, 0, z) : z \in \mathbb{R}\}$ by*

$$\begin{aligned} (x, y, z) &= f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\ &= \left(\frac{(a^2+b^2+1)(a+d)}{(a+d)^2+(b-c)^2}, \frac{(a^2+b^2+1)(b-c)}{(a+d)^2+(b-c)^2}, \frac{ac+bd}{\sqrt{(a+d)^2+(b-c)^2}}\right). \end{aligned} \tag{3.3}$$

Then, f is bijective, and f^{-1} is given by

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= f^{-1}(x, y, z) \\ &= \begin{pmatrix} x - \dfrac{z}{r}y & y + \dfrac{z}{r}x \\ -\dfrac{z^2+1}{r^2}y + \dfrac{z}{r}x & \dfrac{z^2+1}{r^2}x + \dfrac{z}{r}y \end{pmatrix}, \end{aligned} \tag{3.4}$$

where $r = \sqrt{x^2 + y^2}$.

Proof. Equation 3.3 is derived from direct calculations. By Equation 3.1,

$$a + d = 2\cosh s \cos\varphi, \quad b - c = 2\cosh s \sin\varphi, \ (a+d)^2 + (b-c)^2 = 4\cosh^2 s,$$

$$a^2 + b^2 + 1 = 2\cosh s(\cosh s + \sinh s \cos\theta), \ ac + bd = 2\cosh s \ \sinh s \sin\theta.$$

Using equations above and taking account of Equation 3.2, one obtains Equation 3.3.
To obtain Equation 3.4, let

$$g(x, y, z) = \begin{pmatrix} x - \dfrac{z}{r}y & y + \dfrac{z}{r}x \\ -\dfrac{z^2+1}{r^2}y + \dfrac{z}{r}x & \dfrac{z^2+1}{r^2}x + \dfrac{z}{r}y \end{pmatrix}$$

be a map from $\mathbb{R}^3 - \{(0, 0, z) : z \in \mathbb{R}\}$ to $SL(2, \mathbb{R})$. It is easy to check that the determinant of the matrix above is equal to 1. And it is not hard to check that $f \circ g = id$. This completes the proof. □

We can also derive Equation 3.4 with geometric considerations. Assume that $A_0 = (x, y, z)$ is not on the z-axis. As in Figure 2, let $H = (x, y)$ be the foot of the perpendicular line through A_0 perpendicular to the xy-plane. Let $P_1 = (x, y)/r$ be a point on the unit circle, and let $P_2 = (-y, x)/r$ be the rotated point of P_1 around O with angle $\pi/2$. Then, $\overrightarrow{OA} = \overrightarrow{OH} + z\overrightarrow{OP_2}$, hence,

$$(a, b) = \left(x - \frac{z}{r}y, y + \frac{z}{r}x\right).$$

In the next, we shall calculate the coordinates of T from $A_0 = (x, y, z)$. Since T is on the line OH, we can set $T = s(x, y)$ with a positive number s. The equation $AT^2 + 1 = OT^2$ implies that

$$\left(x - \frac{z}{r}y - sx\right)^2 + \left(y + \frac{z}{r}x - sy\right)^2 + 1 = s^2(x^2 + y^2).$$

So, we get

$$s = \frac{r^2 + z^2 + 1}{2r^2},$$

thus,

$$T = \frac{r^2 + z^2 + 1}{2r^2}(x, y).$$

By $\overrightarrow{OA} + \overrightarrow{OB'} = 2\overrightarrow{OT}$,

$$B' = \left(\frac{z^2 + 1}{r^2}x + \frac{z}{r}y, \frac{z^2 + 1}{r^2}y - \frac{z}{r}x\right).$$

Rotating B' around O with angle $\pi/2$, we get

$$(c, d) = \left(-\frac{z^2 + 1}{r^2}y + \frac{z}{r}x, \frac{z^2 + 1}{r^2}x + \frac{z}{r}y\right).$$

Roughly speaking, Equation 3.3 indicates that each axis has geometric meaning. The x-axis is the axis of trace of the matrix, the yz-plane corresponds to the subset of matrices with zero trace. The y-axis is the axis of asymmetry of the matrix, the xz-plane corresponds to the subset of symmetric matrices. The z-axis is the axis of angle, the xy-plane corresponds to the subset of rectangles. If $z > 0$, $\angle AOB$ is acute, and if $z < 0$, $\angle AOB$ is obtuse. In addition, by Equation 3.4, it is easy to check that $a^2 + b^2 = c^2 + d^2$ if and only if $x^2 + y^2 - z^2 = 1$. Hence, hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ corresponds to the subset of rhombuses such that $OA = OB$. Therefore, the unit circle on the xy-plane which is the intersection of $x^2 + y^2 - z^2 = 1$ and the xy-plane corresponds to the subset of squares, that is, the subgroup $SO(2)$ defined as ([1] p.34, [2] p.67, [3] p.19)

$$SO(2) = \left\{ M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SL(2, \mathbb{R}) \;\middle|\; \theta \in [0, 2\pi) \right\}.$$

The subgroup $SO(1, 1)$ defined as ([1] p.43, [2] p.70)

$$SO(1,1) = \left\{ M = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}, \begin{pmatrix} -\cosh s & \sinh s \\ \sinh s & -\cosh s \end{pmatrix} \in SL(2, \mathbb{R}) \;\middle|\; s \in \mathbb{R} \right\}$$

corresponds to hyperbola $\{x^2 - z^2 = 1, y = 0\}$, and the set of diagonal matrices corresponds to the x-axis except for the origin $O = (0, 0, 0)$.

4 Visualization of the exponential map

With the three dimensional model, we can visualize $\mathfrak{sl}(2, \mathbb{R})$ as tangent vectors around $(1, 0, 0)$.

For the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ defined as ([1] , p.112)

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ X = \begin{pmatrix} \alpha & \beta+\gamma \\ -\beta+\gamma & -\alpha \end{pmatrix} \;\middle|\; \alpha,\beta,\gamma \in \mathbb{R} \right\},$$

the exponential map e^{tX} is given as

$$e^{tX} = I_2 + tX + O(t^2) = \begin{pmatrix} 1+\alpha t & (\beta+\gamma)t \\ (-\beta+\gamma)t & 1-\alpha t \end{pmatrix} + O(t^2) \quad (t \to 0).$$

Set $e^{tX} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then,

$$a+d = 2+O(t^2), \quad b-c = 2\beta t + O(t^2), \quad a^2+b^2+1 = 2+2\alpha t + O(t^2),$$

$$ac+bd = 2\gamma t + O(t^2), \quad (a+d)^2+(b-c)^2 = 4+O(t^2) \qquad (t \to 0). \tag{4.1}$$

In the following discussion, we will investigate the tangent vector of the exponential map at the point $(1,0,0)$. To do this, we prepare two lemmas.

Lemma 4.1. *Let $h(t)$ and $k(t)$ be analytic functions in a neighborhood of $t=0$ such that*

$$\begin{aligned} h(t) &= h_0 + h_1 t + O(t^2) \quad (t \to 0), \\ k(t) &= k_0 \qquad + O(t^2) \quad (t \to 0) \end{aligned}$$

where $h_0, h_1, k_0 \in \mathbb{R}$ and $k_0 \neq 0$ and assume that $k(t) \neq 0$ in the neighborhood of $t=0$. Then,

$$\frac{h(t)}{k(t)} = \frac{h_0}{k_0} + \frac{h_1}{k_0} t + O(t^2) \quad (t \to 0). \tag{4.2}$$

Proof. By the assumptions,

$$h(0) = h_0, \quad h'(0) = h_1, \quad k(0) = k_0, \quad k'(0) = 0.$$

The function $\varphi(t) = \frac{h(t)}{k(t)}$ is an analytic function in the neighborhood of $t=0$. Hence by Taylor's theorem,

$$\varphi(t) = \varphi(0) + \varphi'(0)t + O(t^2) \quad (t \to 0).$$

Since

$$\varphi(0) = \frac{h(0)}{k(0)} = \frac{h_0}{k_0}$$

and

$$\varphi'(0) = \frac{h'(0)\cdot k(0) - h(0)\cdot k'(0)}{k(0)^2} = \frac{h_1 \cdot k_0 - h_0 \cdot 0}{k_0^2} = \frac{h_1}{k_0},$$

we get Equation 4.2. This completes the proof. □

Lemma 4.2. *Let $k(t)$ be an analytic function in a neighborhood of $t=0$ such that*

$$k(t) = k_0 + O(t^2) \quad (t \to 0)$$

where k_0 is a positive number and assume that $k(t) > 0$ in the neighborhood of $t=0$. Then,

$$\sqrt{k(t)} = \sqrt{k_0} + O(t^2) \quad (t \to 0). \tag{4.3}$$

Proof. Set $\psi(t) = \sqrt{k(t)}$. Then,

$$\psi(0) = \sqrt{k(0)} = \sqrt{k_0}, \quad \text{and} \quad \psi'(0) = \frac{1}{2}k(0)^{-\frac{1}{2}} \cdot k'(0) = \frac{1}{2}\frac{1}{\sqrt{k_0}} \cdot 0 = 0.$$

By Taylor's theorem,

$$\psi(t) = \psi(0) + \psi'(0)t + O(t^2) = \sqrt{k_0} + O(t^2) \qquad (t \to 0).$$

Hence, we get Equation 4.3. This completes the proof. □

Proposition 4.3. *Let a, b, c, and d be functions of t satisfying Equations 4.1:*

$$a + d = 2 + O(t^2), \quad b - c = 2\beta t + O(t^2), \quad a^2 + b^2 + 1 = 2 + 2\alpha t + O(t^2),$$

$$ac + bd = 2\gamma t + O(t^2), \quad (a + d)^2 + (b - c)^2 = 4 + O(t^2) \qquad (t \to 0).$$

Then, the functions x, y, and z of t defined by Equation 3.3

$$(x, y, z) = \left(\frac{(a^2 + b^2 + 1)(a + d)}{(a + d)^2 + (b - c)^2}, \frac{(a^2 + b^2 + 1)(b - c)}{(a + d)^2 + (b - c)^2}, \frac{ac + bd}{\sqrt{(a + d)^2 + (b - c)^2}} \right)$$

are represented by the following Taylor series:

$$\begin{aligned} x &= 1 + \alpha t + O(t^2) \qquad (t \to 0), \\ y &= \beta t + O(t^2) \qquad (t \to 0), \\ z &= \gamma t + O(t^2) \qquad (t \to 0). \end{aligned} \tag{4.4}$$

Proof. By the assumptions,

$$\begin{aligned} (a^2 + b^2 + 1)(a + d) &= \left(2 + 2\alpha t + O(t^2)\right)\left(2 + O(t^2)\right) = 4 + 4\alpha t + O(t^2), \\ (a + d)^2 + (b - c)^2 &= 4 + O(t^2). \end{aligned}$$

Then, by Lemma 4.1,

$$\begin{aligned} x &= \frac{4}{4} + \frac{4\alpha}{4}t + O(t^2) \\ &= 1 + \alpha t + O(t^2), \end{aligned}$$

which is the Taylor series for x. In a similar way,

$$(a^2 + b^2 + 1)(b - c) = \left(2 + 2\alpha t + O(t^2)\right)\left(2\beta t + O(t^2)\right) = 0 + 4\beta t + O(t^2),$$

hence, using Lemma 4.1 again,

$$\begin{aligned} y &= \frac{0}{4} + \frac{4\beta}{4}t + O(t^2) \\ &= \beta t + O(t^2), \end{aligned}$$

which is the Taylor series for y. As for z, by Lemma 4.2,

$$\sqrt{(a + d)^2 + (b - c)^2} = 2 + O(t^2).$$

Therefore, using Lemma 4.1 again,

$$\begin{aligned} z &= \frac{0}{2} + \frac{2\gamma}{2}t + O(t^2) \\ &= \gamma t + O(t^2). \end{aligned}$$

This completes the proof. □

Proposition 4.3 implies that

$$(x(t), y(t), z(t)) = (1, 0, 0) + (\alpha, \beta, \gamma)t + O(t^2) \quad (t \to 0).$$

Therefore, the orbit of the exponential map is tangent to the vector (α, β, γ) at the point $(1, 0, 0)$.

In the next, we will investigate the image of the exponential map. Since

$$X^2 = -(\det X) I_2,$$

the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is classified into three types:

$$X = \begin{pmatrix} \alpha & \beta + \gamma \\ -\beta + \gamma & -\alpha \end{pmatrix} \longrightarrow \begin{cases} \text{timelike:} & \alpha^2 - \beta^2 + \gamma^2 = -1, & X^2 = -I_2 \\ \text{lightlike:} & \alpha^2 - \beta^2 + \gamma^2 = 0, & X^2 = O \\ \text{spacelike:} & \alpha^2 - \beta^2 + \gamma^2 = +1, & X^2 = +I_2 \end{cases},$$

where X is normalized for simplicity. For each type, the exponential map is given as the followings:

$$e^{tX} = \begin{cases} (\cos t) I_2 + (\sin t) X, & \mathrm{Tr}(e^{tX}) = 2\cos t \in [-2, 2] \\ I_2 + tX, & \mathrm{Tr}(e^{tX}) \equiv 2 \\ (\cosh t) I_2 + (\sinh t) X, & \mathrm{Tr}(e^{tX}) = 2\cosh t \in [2, +\infty) \end{cases}. \tag{4.5}$$

For each type, using Equation 3.3, we can see the exponential orbits of (x, y, z) with parameter t as shown in Figure 3. In Figure 3, yellow curve is an exponential orbit with lightlike vector which

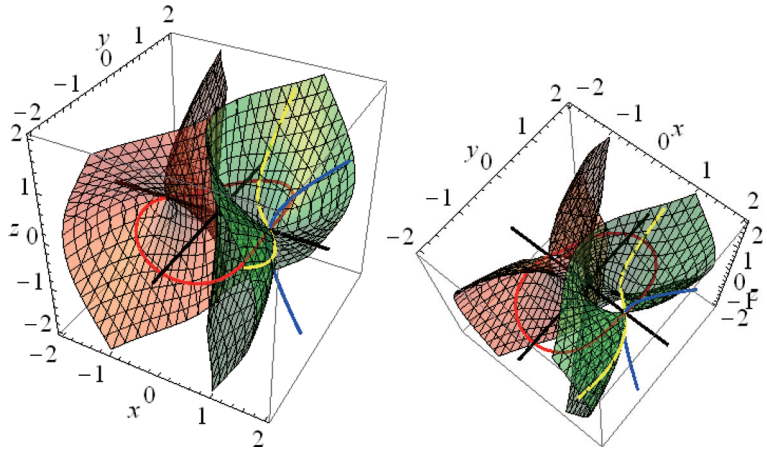

Figure 3: Three types of the exponential orbits.

lies on the green surface. Red curve is an exponential orbit with timelike vector which is a closed curve sandwiched between the green surface and the orange surface. Blue curve is an exponential orbit with spacelike vector which is on the same side of the green surface as the positive x-axis.

By Equation 3.4, the subset of matrices with trace 2 corresponds to the surface $x(r^2+z^2+1) = 2r^2$. By Equation 4.5, any orbit with lightlike vector is on this surface. This surface has a cone-like shape around $(1,0,0)$ as in Figure 3(right) because the lightlike vectors (α,β,γ) satisfy $\alpha^2-\beta^2+\gamma^2=0$.

By Equation 4.5, any orbit with timelike vector is in the region sandwiched between two surfaces; the surface with trace 2 and the surface with trace -2:

$$\left\{(x,y,z)\in\mathbb{R}^3-\{(0,0,z):z\in\mathbb{R}\} \mid -2r^2<x(r^2+z^2+1)<2r^2\right\}\cup\{(\pm1,0,0)\}.$$

Note that any orbit with timelike vector is a closed curve symmetric with respect to the z-axis. This fact comes from Equations 3.3 and 4.5 since $e^{(t+\pi)X}=-e^{tX}$ and

$$f\left(-\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(-\frac{(a^2+b^2+1)(a+d)}{(a+d)^2+(b-c)^2}, -\frac{(a^2+b^2+1)(b-c)}{(a+d)^2+(b-c)^2}, \frac{ac+bd}{\sqrt{(a+d)^2+(b-c)^2}}\right).$$

In particular, any orbit with timelike vector passes through the point $(-1,0,0)$ which corresponds to the matrix $-I_2$.

On the other hand, any orbit with spacelike vector is in the region defined as follows:

$$\left\{(x,y,z)\in\mathbb{R}^3 \mid 2r^2<x(r^2+z^2+1)\right\}\cup\{(+1,0,0)\},$$

because by Equation 4.5 the trace of the exponential map is greater than 2 unless $t=0$.

According to the set (1.1) mentioned in Section 1, the image of the exponential map in this model is given as

$$\left\{(x,y,z)\in\mathbb{R}^3-\{(0,0,z):z\in\mathbb{R}\} \mid -2r^2<x(r^2+z^2+1)\right\}\cup\{(-1,0,0)\}.$$

In this way, we can see the image of the exponential map which is bounded by the surface $x(r^2+z^2+1)=-2r^2$, that is, the half-turned surface of the surface $x(r^2+z^2+1)=2r^2$ around the z-axis.

5 Visualization of non-commutativity

With the three dimensional model, we can also visualize the difference between matrix multiplications AB and BA for $A,B\in SL(2,\mathbb{R})$. For instance, let A and B be elements in $SL(2,\mathbb{R})$ such that

$$A=\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},$$

$$B=\begin{pmatrix} \cosh s\cos\varphi+\sinh s\cos(\varphi+\theta) & \cosh s\sin\varphi+\sinh s\sin(\varphi+\theta) \\ -\cosh s\sin\varphi+\sinh s\sin(\varphi+\theta) & \cosh s\cos\varphi-\sinh s\cos(\varphi+\theta) \end{pmatrix}.$$

In the three dimensional model, regarding $\mathbb{R}^3$ as $\mathbb{C}\times\mathbb{R}$, and using Equations 3.1 and 3.2, the corresponding points of A and B, respectively, are

$$(x_A+iy_A,z_A)=(e^{i\alpha},0),\quad (x_B+iy_B,z_B)=\left((\cosh s+\sinh s\cos\theta)e^{i\varphi},\sinh s\sin\theta\right).$$

On the other hand, since

$$AB=\begin{pmatrix} \cosh s\cos(\varphi+\alpha)+\sinh s\cos(\varphi+\theta-\alpha) & \cosh s\sin(\varphi+\alpha)+\sinh s\sin(\varphi+\theta-\alpha) \\ -\cosh s\sin(\varphi+\alpha)+\sinh s\sin(\varphi+\theta-\alpha) & \cosh s\cos(\varphi+\alpha)-\sinh s\cos(\varphi+\theta-\alpha) \end{pmatrix},$$

$$BA = \begin{pmatrix} \cosh s\cos(\varphi+\alpha)+\sinh s\cos(\varphi+\alpha+\theta) & \cosh s\sin(\varphi+\alpha)+\sinh s\sin(\varphi+\alpha+\theta) \\ -\cosh s\sin(\varphi+\alpha)+\sinh s\sin(\varphi+\alpha+\theta) & \cosh s\cos(\varphi+\alpha)-\sinh s\cos(\varphi+\alpha+\theta) \end{pmatrix},$$

the corresponding points of AB and BA, respectively, are

$$\begin{aligned} (x_{AB}+iy_{AB}, z_{AB}) &= \Big((\cosh s+\sinh s\cos(\theta-2\alpha))e^{i(\varphi+\alpha)}, \sinh s\sin(\theta-2\alpha)\Big), \\ (x_{BA}+iy_{BA}, z_{BA}) &= \Big((\cosh s+\sinh s\cos\theta)e^{i(\varphi+\alpha)}, \sinh s\sin\theta\Big). \end{aligned}$$

When we change the value of the parameter α of A from 0 to 2π, both orbits of AB and BA in the three dimensional model lie on the torus $(r-\cosh s)^2+z^2=\sinh^2 s$. The orbit of BA is a circle, that is, a longitude of the torus. On the other hand, the orbit of AB is the (1,2)-torus knot which winds 2 times around the unit circle on the xy-plane, and 1 time around the z-axis as shown in Figure 4.

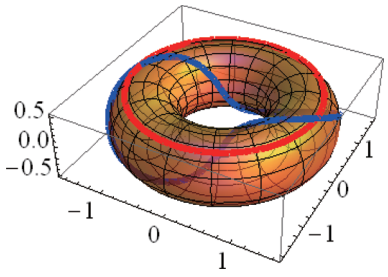

Figure 4: Visualization of non-commutativity.

In this way, the three dimensional model is also useful to visualize non-commutativity of matrix multiplication.

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