

Multiple Comparison for Checking Correlations among Several Normal Random Variables

by

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(Received on Sept. 17, 2014; accepted on Dec. 3, 2014)

Abstract

In this study we discuss a multiple comparison for checking correlations among several normal random variables. Specifically, we construct two kinds of multiple comparison procedures using t -distribution based on sample correlation coefficient for pairwise comparison. One is a single step procedure based on Bonferroni's inequality. The other is a sequentially rejective step down procedure based on Holm's method. We compare two procedures in terms of the power of the test in a simulation study. Finally, we give an example to illustrate our procedures.

Key words: Bonferroni's inequality, Holm's method, Power of the test, t -distribution

1 Introduction

To test whether two normal random variables are independent, we use the t -distribution based on the sample correlation coefficient. To test whether several normal random variables are independent, we use the χ^2 -distribution which is the asymptotic distribution of the likelihood ratio test statistic (cf. Anderson (2003) [1]). If they are not independent, we occasionally want to know detailed correlations among them. Imada (2014) [4] discussed multiple comparisons for checking the correlation between a specified variable and each of other valuables using t -statistic for pairwise comparison. Specifically, Imada (2014) [4] proposed a single step procedure and a sequentially rejective step down procedure and compared them in terms of power of the test through simulations.

The aim of this study is to discuss multiple comparisons for checking detailed correlations among several normal random variables. Specifically, we construct a single step procedure and a

sequentially rejective step down procedure using t -statistic for pairwise comparison. In the single step procedure we determine the critical value for a specified significance level using Bonferroni's inequality. In the step down procedure we determine the critical value of each step for a specified significance level using Holm's method. We compare two procedures in terms of power of the test through simulations. Finally, we give an example to illustrate our procedures.

2 Multiple comparison for checking correlations among several normal random variables

Assume p -dimensional random variable vector $\mathbf{X} = (X_1, X_2, \dots, X_p)'$ is distributed according to multivariate normal $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where a mean vector $\boldsymbol{\mu}$ and a covariance matrix $\boldsymbol{\Sigma}$ are unknown. We want to check detailed correlations among $X_1, X_2, \dots, X_p$. Letting ρ_{ij} be the correlation coefficient between X_i and X_j , we set up a null hypothesis and its alternative hypothesis as

$$H_{ij} : \rho_{ij} = 0 \text{ vs. } H_{ij}^A : \rho_{ij} \neq 0. \quad (2.1)$$

We consider the simultaneous test for H_{ij} ($i < j$). Let $\mathbf{X}_i = (X_{i1}, X_{i2}, \dots, X_{ip})'$ ($i = 1, 2, \dots, n$) be a sample from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. For testing (2.1) we use the statistic

$$t_{ij} = \frac{\sqrt{n-2}r_{ij}}{\sqrt{1-r_{ij}^2}}$$

where

$$r_{ij} = \frac{\sum_{l=1}^n (X_{li} - \bar{X}_i)(X_{lj} - \bar{X}_j)}{\sqrt{\sum_{l=1}^n (X_{li} - \bar{X}_i)^2} \sqrt{\sum_{l=1}^n (X_{lj} - \bar{X}_j)^2}}$$

is the sample correlation coefficient between X_i and X_j . t_{ij} is distributed according to t -distribution with $n-2$ degrees of freedom under H_{ij} . We express the distribution as $t_{ij} \sim t_{n-2}$.

3 Single step procedure

3.1 Construction of procedure

In the single step procedure for H_{ij} ($i < j$) we specify a critical value c and judge as follows.

Case 1. If $|t_{ij}| > c$, we reject H_{ij} .

Case 2. If $|t_{ij}| \leq c$, we retain H_{ij} .

We determine c so that the maximum type I FWE (familywise error rate) may not be larger than a specified significance level α . Therefore, we should determine c so that

$$P(\max_{i < j} |t_{ij}| > c) = \alpha$$

under the condition that all of H_{ij} 's are true. Since t_{ij} 's are not independent and it is difficult to derive their joint distribution, we determine a conservative critical value. By Bonferroni's inequality

$$P(\max_{i < j} |t_{ij}| > c) \leq \sum_{i < j} P(|t_{ij}| > c). \quad (3.1)$$

If we determine c so that

$$P(|t_{n-2}| > c) = \frac{\alpha}{p(p-1)/2}, \quad (3.2)$$

we obtain

$$P(\max_{2 \leq i \leq p} |t_{1i}| > c) \leq \alpha$$

by (3.1).

3.2 Power of the test

Next, we consider the power of the test. There are several kinds of definitions of the power of multiple comparison procedures, which are explained in detail in Nagata and Yoshida (1997) [5]. In this study we focus on the all-pairs power defined by Ramsey (1978) [6].

Let $\mathbf{S}$ be a subset of $\{(i, j) | i \text{ and } j \text{ are integers satisfying } 1 \leq i < j \leq p\}$. We assume

$$\rho_{ij} \neq 0 \text{ for } (i, j) \in \mathbf{S}, \quad \rho_{ij} = 0 \text{ for } (i, j) \notin \mathbf{S}. \quad (3.3)$$

The all-pairs power of (3.3) is

$$P(\min_{(i,j) \in \mathbf{S}} |t_{ij}| > c). \quad (3.4)$$

When we calculate the power, we specify the value of ρ_{ij} for all $(i, j) \in \mathbf{S}$. However, it is difficult to calculate (3.4). Therefore, we specify Σ and calculate the approximate value of (3.4) by Monte Carlo simulation generating random numbers from $N_p(\boldsymbol{\mu}, \Sigma)$. Here we can assume $\boldsymbol{\mu} = \mathbf{0}$ without loss of generality.

Specify the total number of experiments N and let $R = 0$. The algorithm for obtaining the approximate value of (3.4) by Monte Carlo simulation is given as follows.

Step 1.

Generate random numbers $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ from $N_p(\mathbf{0}, \Sigma)$ and calculate t_{ij} for all $(i, j) \in \mathbf{S}$.

Step 2.

Case 1. $\min_{(i,j) \in \mathbf{S}} |t_{ij}| > c \Rightarrow R = R + 1,$

Case 2. $\min_{(i,j) \in \mathbf{S}} |t_{ij}| \leq c \Rightarrow R = R.$

Repeating these steps with N times, we obtain the approximate value R/N of (3.4).

4 Sequentially rejective step down procedure

In this section we discuss a sequentially rejective step down procedure called Holm's method.

4.1 Construction of procedure

We determine c_q for $q = 1, 2, \dots, p(p-1)/2$ so that

$$P(|t_{n-2}| > c_q) = \frac{\alpha}{q}. \quad (4.1)$$

Then $c_{p(p-1)/2} > c_{p(p-1)/2-1} > \dots > c_1$. Arranging $|t_{ij}|$'s in order of a size of value, assume

$$|t_{(1)}| \leq |t_{(2)}| \leq \dots \leq |t_{(p(p-1)/2)}|. \quad (4.2)$$

$H_{(1)}, H_{(2)}, \dots, H_{(p(p-1)/2)}$ denotes hypotheses corresponding to $t_{(1)}, t_{(2)}, \dots, t_{(p(p-1)/2)}$. We test

H_{ij} 's sequentially as follows.

Step 1.

Case 1. If $|t_{(p(p-1)/2)}| \leq c_{p(p-1)/2}$, we retain all H_{ij} 's and stop the test.

Case 2. If $|t_{(p(p-1)/2)}| > c_{p(p-1)/2}$, we reject $H_{(p(p-1)/2)}$ and go to the next step.

Step 2.

Case 1. If $|t_{(p(p-1)/2-1)}| \leq c_{p(p-1)/2-1}$, we retain $H_{(1)}, H_{(2)}, \dots, H_{(p(p-1)/2-1)}$ and stop the test.

Case 2. If $|t_{(p(p-1)/2-1)}| > c_{p(p-1)/2-1}$, we reject $H_{(p(p-1)/2-1)}$ and go to the next step.

$\vdots$

We repeat similar judgments till up to Step $p(p-1)/2$.

Imada and Douke (2007b) [3] indicated that the maximum type I FWE of Holm's method is not greater than α .

4.2 Power of the test

Next, we consider the power of the test. We consider the power under (3.3). The power is the probability that all $H_{i,j}$ ($(i, j) \in \mathbf{S}$) are rejected in the step down test. We calculate it by Monte Carlo simulation generating random numbers from $N_p(\mathbf{0}, \mathbf{\Sigma})$ for a specified $\mathbf{\Sigma}$.

Specify the total number of experiments N and let $R = 0$. The algorithm for obtaining the approximate value of the power is given as follows.

Step 1.

Generate random numbers $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ from $N_p(\mathbf{0}, \mathbf{\Sigma})$ and calculate all t_{ij} 's.

Step 2.

We carry out the step down test using t_{ij} 's.

Case 1. If t_{ij} for each $(i, j) \in \mathbf{S}$ is rejected in the step down test, $R = R + 1$.

Case 2. Otherwise, let $R = R$.

Repeating these steps with N times, we obtain the approximate value R/N of the power.

Imada and Douke (2007a) [2] indicated that the power of the step down procedure is uniformly higher than that of the single step procedure.

5 Simulation results

In this section we give some simulation results intended to compare the single step procedure and the step down procedure. Here SS denotes the single step procedure and SD denotes the step down procedure.

Let $\alpha = 0.05$ and $n = 20, 30$. Let the number of repetition of experiment $N = 100,000$ for Monte Carlo simulation.

First, let $p = 4$. When $n = 20$, the critical value of SS determined by (3.2) is $c = 2.963$ and the critical values of SD determined by (4.1) are

$$c_6 = 2.963, c_5 = 2.879, c_4 = 2.775, c_3 = 2.640, c_2 = 2.446, c_1 = 2.101.$$

When $n = 30$, the critical value of SS determined by (3.2) is $c = 2.839$ and the critical values of

SD determined by (4.1) are

$$c_6 = 2.839, c_5 = 2.764, c_4 = 2.670, c_3 = 2.547, c_2 = 2.369, c_1 = 2.049.$$

Next, let $p = 5$. When $n = 20$, the critical value of SS determined by (3.2) is $c = 3.197$ and the critical values of SD determined by (4.1) are

$$c_{10} = 3.197, c_9 = 3.149, c_8 = 3.095, c_7 = 3.034, c_6 = 2.963,$$

$$c_5 = 2.879, c_4 = 2.775, c_3 = 2.640, c_2 = 2.446, c_1 = 2.101.$$

When $n = 30$, the critical value of SS determined by (3.2) is $c = 3.047$ and the critical values of SD determined by (4.1) are

$$c_{10} = 3.047, c_9 = 3.005, c_8 = 2.957, c_7 = 2.903, c_6 = 2.839,$$

$$c_5 = 2.764, c_4 = 2.670, c_3 = 2.547, c_2 = 2.369, c_1 = 2.049.$$

Next, we consider the power of the test. First, let $p = 4$. To calculate the power by Monte Carlo simulation we set up three types of covariance matrices as follows.

$$\mathbf{\Sigma}_1 = \begin{pmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix}, \quad \mathbf{\Sigma}_2 = \begin{pmatrix} 1 & \rho & \rho & 0 \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ 0 & \rho & \rho & 1 \end{pmatrix}, \quad \mathbf{\Sigma}_3 = \begin{pmatrix} 1 & \rho & 0 & 0 \\ \rho & 1 & \rho & 0 \\ 0 & \rho & 1 & \rho \\ 0 & 0 & \rho & 1 \end{pmatrix}.$$

$\mathbf{\Sigma}_1$ means that for each pair (X_i, X_j) they are related. $\mathbf{\Sigma}_3$ is a tridiagonal matrix and means that X_i and X_{i+1} for $i = 1, 2, 3$ are related. $\mathbf{\Sigma}_2$ is the intermediate type compared to $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_3$. Table 1 gives the power of SS and SD for $\mathbf{\Sigma} = \mathbf{\Sigma}_1$ where $\rho = 0.1, 0.2, \dots, 0.9$. Tables 2,3 give the power of SS and SD for $\mathbf{\Sigma} = \mathbf{\Sigma}_2, \mathbf{\Sigma}_3$, respectively, where $\rho = 0.1, 0.2, \dots, 0.6$. We choose ρ so that $\rho > 0$ and $\mathbf{\Sigma}$ may be positive definite. Tables show that the power increases as ρ increases for $\mathbf{\Sigma} = \mathbf{\Sigma}_1, \mathbf{\Sigma}_2, \mathbf{\Sigma}_3$. The power using $n = 30$ is uniformly higher than that using $n = 20$ in each case. SD is uniformly more powerful compared to SS in each case.

Next, let $p = 5$. To calculate the power by Monte Carlo simulation we set up four types of covariance matrices as follows.

$$\mathbf{\Sigma}_1 = \begin{pmatrix} 1 & \rho & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho & \rho \\ \rho & \rho & 1 & \rho & \rho \\ \rho & \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & \rho & 1 \end{pmatrix}, \quad \mathbf{\Sigma}_2 = \begin{pmatrix} 1 & \rho & \rho & \rho & 0 \\ \rho & 1 & \rho & \rho & \rho \\ \rho & \rho & 1 & \rho & \rho \\ \rho & \rho & \rho & 1 & \rho \\ 0 & \rho & \rho & \rho & 1 \end{pmatrix},$$

$$\mathbf{\Sigma}_3 = \begin{pmatrix} 1 & \rho & \rho & 0 & 0 \\ \rho & 1 & \rho & \rho & 0 \\ \rho & \rho & 1 & \rho & \rho \\ 0 & \rho & \rho & 1 & \rho \\ 0 & 0 & \rho & \rho & 1 \end{pmatrix}, \quad \mathbf{\Sigma}_4 = \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & \rho & 0 & 0 \\ 0 & \rho & 1 & \rho & 0 \\ 0 & 0 & \rho & 1 & \rho \\ 0 & 0 & 0 & \rho & 1 \end{pmatrix}.$$

$\mathbf{\Sigma}_1$ means that for each pair (X_i, X_j) they are related. $\mathbf{\Sigma}_4$ is a tridiagonal matrix and means that

X_i and X_{i+1} for $i = 1, 2, 3, 4$ are related. Σ_2 and Σ_3 are the intermediate types compared to Σ_1 and Σ_4 . Table 4 gives the power of SS and SD for $\Sigma = \Sigma_1$ where $\rho = 0.1, 0.2, \dots, 0.9$. Tables 5, 6 give the power of SS and SD for $\Sigma = \Sigma_2, \Sigma_3$, respectively, where $\rho = 0.1, 0.2, \dots, 0.6$. Table 7 give the power of SS and SD for $\Sigma = \Sigma_4$, respectively, where $\rho = 0.1, 0.2, 0.3, 0.4, 0.5$. The aspects of variation of the power corresponding to the various situations are similar to those for $p = 4$.

Table 1 : Power comparison for $\Sigma = \Sigma_1$ ($p = 4$)

	ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$n = 20$	SS	0.000	0.000	0.000	0.006	0.044	0.206	0.561	0.907	0.999
	SD	0.000	0.000	0.005	0.042	0.186	0.489	0.818	0.980	1.000
$n = 30$	SS	0.000	0.000	0.002	0.028	0.178	0.543	0.890	0.995	1.000
	SD	0.000	0.001	0.019	0.134	0.447	0.808	0.975	0.999	1.000

Table 2 : Power comparison for $\Sigma = \Sigma_2$ ($p = 4$)

	ρ	0.1	0.2	0.3	0.4	0.5	0.6
$n = 20$	SS	0.000	0.000	0.000	0.004	0.038	0.199
	SD	0.000	0.000	0.002	0.020	0.111	0.373
$n = 30$	SS	0.000	0.000	0.002	0.025	0.170	0.545
	SD	0.000	0.001	0.009	0.079	0.334	0.726

Table 3 : Power comparison for $\Sigma = \Sigma_3$ ($p = 4$)

	ρ	0.1	0.2	0.3	0.4	0.5	0.6
$n = 20$	SS	0.000	0.000	0.000	0.005	0.039	0.202
	SD	0.000	0.000	0.001	0.009	0.061	0.268
$n = 30$	SS	0.000	0.000	0.003	0.031	0.192	0.587
	SD	0.000	0.000	0.006	0.050	0.256	0.662

Table 4 : Power comparison for $\Sigma = \Sigma_1$ ($p = 5$)

	ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$n = 20$	SS	0.000	0.000	0.000	0.000	0.009	0.084	0.376	0.826	0.996
	SD	0.000	0.000	0.001	0.014	0.099	0.365	0.750	0.969	1.000
$n = 30$	SS	0.000	0.000	0.000	0.004	0.064	0.346	0.793	0.987	1.000
	SD	0.000	0.000	0.004	0.060	0.313	0.725	0.960	0.999	1.000

Table 5 : Power comparison for $\Sigma = \Sigma_2$ ($p = 5$)

	ρ	0.1	0.2	0.3	0.4	0.5	0.6
$n = 20$	SS	0.000	0.000	0.000	0.000	0.008	0.083
	SD	0.000	0.000	0.000	0.006	0.054	0.266
$n = 30$	SS	0.000	0.000	0.000	0.004	0.060	0.347
	SD	0.000	0.000	0.001	0.031	0.217	0.631

Table 6 : Power comparison for $\Sigma = \Sigma_3$ ($p = 5$)

ρ		0.1	0.2	0.3	0.4	0.5	0.6
$n = 20$	SS	0.000	0.000	0.000	0.000	0.005	0.065
	SD	0.000	0.000	0.000	0.001	0.019	0.150
$n = 30$	SS	0.000	0.000	0.000	0.002	0.049	0.335
	SD	0.000	0.000	0.000	0.012	0.122	0.508

Table 7 : Power comparison for $\Sigma = \Sigma_4$ ($p = 5$)

ρ		0.1	0.2	0.3	0.4	0.5
$n = 20$	SS	0.000	0.000	0.000	0.000	0.005
	SD	0.000	0.000	0.000	0.005	0.053
$n = 30$	SS	0.000	0.000	0.000	0.004	0.063
	SD	0.000	0.000	0.002	0.036	0.232

6 Example

We give an example in order to illustrate our procedures. Assume that 30 pupils in a high school took four kinds of examinations mathematics, physics, chemistry and biology. The scores are denoted by X_1, X_2, X_3, X_4 , respectively. We check correlations among X_1, X_2, X_3, X_4 for $\alpha = 0.05$ using SS and SD. Table 8 gives means of scores of 4 subjects. Tables 9 and 10 give sample correlation coefficients and t -statistics, respectively. By SS we judge only H_{12} is rejected. By SD we judge H_{12} and H_{34} are rejected. The correlation between the score of mathematics and that of physics for individual pupil is detected by SS and SD. However, the correlation between the score of chemistry and that of biology is detected only by SD.

Table 8 : Means of 4 subjects

Mathematics ($\bar{X}_1$)	Physics ($\bar{X}_2$)	Chemistry ($\bar{X}_3$)	Biology ($\bar{X}_4$)
69.982	59.737	74.761	70.042

Table 9 : Sample correlation coefficients

r_{12}	r_{13}	r_{14}	r_{23}	r_{24}	r_{34}
0.579	0.171	0.164	0.102	0.073	0.468

Table 10 : t -statistics

t_{12}	t_{13}	t_{14}	t_{23}	t_{24}	t_{34}
3.757	0.919	0.881	0.542	0.389	2.804

7 Conclusions

In this study we have discussed the multiple comparison procedures for checking correlations among several normal random variables constructed using t -statistic for pairwise comparison. Specifically, we proposed the single step procedure and the sequentially rejective step down procedure. We compared two procedures in terms of power of the test through simulations. The simulation results showed that the power of the sequentially rejective step down procedure is uniformly higher compared to the single step procedure.

There remain some problems to be solved in the future. We should derive the joint distribution of the statistics for pairwise comparisons intended to obtain the critical value satisfying a specified significance level exactly. We should construct other types of multiple comparison procedures for checking correlations among several normal random variables like step up procedure.

Acknowledgements. The authors are deeply grateful to the referee and the editors for their valuable comments and suggestions.

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