

Multivariate One-sided Test for Comparing Two Normal Mean Vectors Based on Likelihood Ratio Test with Unknown Common Covariance Matrix

by

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Abstract

We discuss a certain multivariate one-sided test for comparing two normal mean vectors based on the likelihood ratio test under the assumption that a common covariance matrix is unknown. Since it is difficult to determine the distribution of the likelihood ratio test statistic independently of the unknown covariance matrix, we discuss two approaches to carry out the test for a specified significance level. One is a test by using a conservative critical value derived by the upper bound of the null distribution of the likelihood ratio test statistic for all covariance matrices. The other is a test under the condition that the orthogonal projection of the observed point onto the positive orthant is determined. We give some examples obtained by Monte Carlo simulation regarding critical values and the power of the test intended to compare two tests.

Key words: F -distribution, Monte Carlo simulation, orthogonal projection

1 Introduction

Multivariate one-sided test for testing whether a normal mean vector having nonnegative components is equal to zero is a multivariate analog of the one-sided test for a normal mean. Kudo (1963) [3] derived the null distribution of the likelihood ratio test statistic when the covariance matrix is known. However, if the covariance matrix is unknown, the distribution of the likelihood ratio test statistic depends on the unknown covariance matrix. Perlman (1969) [4] derived an upper bound of the null distribution of the likelihood ratio test statistic for all covariance matrices. We can obtain a conservative critical value for a specified significance level by using the upper bound. Although the likelihood ratio test statistic is determined by obtaining the orthogonal projection

of the observed point onto the positive orthant, Fraser *et al.* (1991) [1] proposed a test under the condition that the orthogonal projection is determined. In Fraser *et al.* (1991) [1]'s test it is possible to determine the critical value satisfying a specified significance level exactly.

On the other hand, we occasionally want to carry out a multivariate one-sided test for comparing two normal mean vectors. Specifically, for two normal mean vectors, assuming that each component of one vector is not smaller than that of the other vector, we test whether the two vectors are different. The aim of this paper is to discuss the multivariate one-sided test for comparing two normal mean vectors based on the likelihood ratio test under the condition that the common covariance matrix is unknown. Since the distribution of the likelihood ratio test statistic depends on the unknown covariance matrix, we discuss two approaches to carry out the test for a specified significance level. First, we construct a conservative critical value by the upper bound of the null distribution of the likelihood ratio test statistic referring to Perlman (1969) [4]. Next, we propose a test under the condition that the orthogonal projection of the observed point onto the positive orthant is determined referring to Fraser *et al.* (1991) [1]. We give some examples obtained by Monte Carlo simulation regarding critical values and power of the test intended to compare two tests.

2 Multivariate one-sided test for a single normal mean vector

First, we discuss a multivariate one-sided test for a single normal mean vector based on the likelihood ratio test.

2.1 Likelihood ratio test

Assume p -dimensional random variable vector $\mathbf{X}$ is distributed according to multivariate normal $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where a mean vector $\boldsymbol{\mu}$ and a covariance matrix $\boldsymbol{\Sigma}$ are unknown. Letting $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_p)'$, assume

$$\mu_1 \geq 0, \mu_2 \geq 0, \dots, \mu_p \geq 0. \quad (2.1)$$

If $\mu_i > 0$ for at least one i ($1 \leq i \leq p$) in (2.1), we express it as $\boldsymbol{\mu} \geq^* \mathbf{0}$. We set up a null hypothesis H_0 and its alternative hypothesis H_1 as

$$H_0 : \boldsymbol{\mu} = \mathbf{0} \quad \text{vs.} \quad H_1 : \boldsymbol{\mu} \geq^* \mathbf{0}. \quad (2.2)$$

We test (2.2) by using a sample $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

By the likelihood ratio test criteria for (2.2) we reject H_0 when

$$\bar{F} = \frac{\|\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})\|_{\mathbf{S}}^2}{1 + \|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})\|_{\mathbf{S}}^2} > c$$

for a specified critical value c . Here

$$\mathbf{Z} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbf{X}_i, \quad \bar{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i, \quad \mathbf{S} = \sum_{i=1}^n (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})'$$

and $\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})$ is the orthogonal projection of $\mathbf{Z}$ onto the positive orthant

$$\boldsymbol{\Theta} = \{\mathbf{x} = (x_1, x_2, \dots, x_p)' \in \mathbf{R}^p | x_1 \geq 0, x_2 \geq 0, \dots, x_p \geq 0\}$$

with regard to Mahalanobis distance $\|\cdot\|_{\mathbf{S}}$ based on $\mathbf{S}$. Specifically, $\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})$ is the nearest point in $\boldsymbol{\Theta}$ from $\mathbf{Z}$ with regard to $\|\cdot\|_{\mathbf{S}}$.

We consider how to determine $\bar{F}$. We consider the decomposition of $\mathbf{Z}$ into two subvectors $\mathbf{Z}_1, \mathbf{Z}_2$ as follows.

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix}. \quad (2.3)$$

Here the dimension of $\mathbf{Z}_1$ is m . Although we should consider all sorts of decompositions of $\mathbf{Z}$ into two subvectors, we consider the simple decomposition like (2.3). The following discussion can be applied to general cases. Let

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix} \quad (2.4)$$

be the decomposition of $\mathbf{S}$ corresponding to (2.3). Therefore, $\mathbf{S}_{11}$ is the $m \times m$ matrix. If

$$\mathbf{Z}_{1 \cdot 2} = \mathbf{Z}_1 - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{Z}_2 > \mathbf{0}, \quad \mathbf{S}_{22}^{-1}\mathbf{Z}_2 \leq \mathbf{0}, \quad (2.5)$$

the orthogonal projection $\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})$ is determined as

$$\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}) = \begin{pmatrix} \mathbf{Z}_{1 \cdot 2} \\ \mathbf{0} \end{pmatrix}. \quad (2.6)$$

Then

$$\|\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2 = \mathbf{Z}_{1 \cdot 2}'\mathbf{S}_{11 \cdot 2}^{-1}\mathbf{Z}_{1 \cdot 2}, \quad \|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2 = \mathbf{Z}_2'\mathbf{S}_{22}^{-1}\mathbf{Z}_2$$

where $\mathbf{S}_{11 \cdot 2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$. Therefore, we obtain

$$\bar{F} = \frac{\mathbf{Z}_{1 \cdot 2}'\mathbf{S}_{11 \cdot 2}^{-1}\mathbf{Z}_{1 \cdot 2}}{1 + \mathbf{Z}_2'\mathbf{S}_{22}^{-1}\mathbf{Z}_2}. \quad (2.7)$$

2.2 Null distribution of the likelihood ratio test statistic

If the number of positive components of $\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})$ is m ,

$$\bar{F} \sim \frac{m}{n-p} F_{m, n-p} \quad (2.8)$$

under H_0 . The distribution (2.8) is determined independently of the event (2.5). Letting $P(m)$ be the probability that the number of positive components of $\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})$ is m , we obtain

$$P(\bar{F} > c) = \sum_{m=1}^p P(m) P\left(\frac{m}{n-p} F_{m, n-p} > c\right).$$

However, $P(m)$ depends on unknown $\boldsymbol{\Sigma}$.

2.3 Determination of conservative critical value

Although we can not determine the critical value c satisfying

$$P(\bar{F} > c) = \alpha$$

under H_0 for a specified significance level α independently of $\boldsymbol{\Sigma}$, we can obtain a conservative critical value by using the upper bound of the distribution of $\bar{F}$ for all covariance matrices under

H_0 derived by Perlman (1969) [4]. The bound is expressed by using F -distributions. Specifically,

$$\max_{\Sigma} P(\bar{F} > c) = \frac{1}{2}P(((p-1)/(n-p))F_{p-1, n-p} > c) + \frac{1}{2}P((p/(n-p))F_{p, n-p} > c).$$

If we determine c so that

$$\frac{1}{2}P(((p-1)/(n-p))F_{p-1, n-p} > c) + \frac{1}{2}P((p/(n-p))F_{p, n-p} > c) = \alpha, \quad (2.9)$$

we obtain

$$P(\bar{F} > c) \leq \alpha,$$

which shows c is a conservative critical value.

Here, the actual Type I error is given by $P(\bar{F} > c)$ under H_0 . To investigate the conservation of c we obtain the approximate value of $P(\bar{F} > c)$ under H_0 for a specified Σ by Monte Carlo simulation. Specify the number of repetition of the experiment N . Let $R = 1$ and $R_1 = 0$. The algorithm of Monte Carlo simulation is given as follows.

Step 1. Generate random numbers $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ from $N_p(\mathbf{0}, \Sigma)$.

Step 2. Determine $\bar{F}$ by $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$. Then

$$\bar{F} > c \Rightarrow R_1 = R_1 + 1, \quad \bar{F} \leq c \Rightarrow R_1 = R_1.$$

Step 3. $R = R + 1$. If $R \leq N$, go to Step 1. Otherwise, stop the experiment and let

$$P^* = \frac{R_1}{N}$$

be an approximate value of $P(\bar{F} > c)$.

Next, we consider the power of the test. Assume that p -dimensional vector $\boldsymbol{\mu}^*$ satisfies $\boldsymbol{\mu}^* \geq^* \mathbf{0}$. The power of the alternative hypothesis $H_1 : \boldsymbol{\mu} = \boldsymbol{\mu}^*$ is $P(\bar{F} > c)$ under H_1 . We calculate its approximate value for a specified Σ by Monte Carlo simulation. Specify the number of repetition of the experiment N . Let $R = 1$ and $R_1 = 0$. The algorithm is given as follows.

Step 1. Generate random numbers $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ from $N_p(\boldsymbol{\mu}^*, \Sigma)$.

Step 2. Determine $\bar{F}$ by $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$. Then

$$\bar{F} > c \Rightarrow R_1 = R_1 + 1, \quad \bar{F} \leq c \Rightarrow R_1 = R_1.$$

Step 3. $R = R + 1$. If $R \leq N$, go to Step 1. Otherwise, stop the experiment and let

$$P^* = \frac{R_1}{N}$$

be an approximate value of $P(\bar{F} > c)$ under H_1 .

2.4 Conditional likelihood ratio test

Next, we discuss a conditional test proposed by Fraser *et al.* (1991) [1]. Specifically, we test (2.2) under the condition that the orthogonal projection $\pi_{\mathcal{S}}(\mathbf{Z}, \boldsymbol{\Theta})$ is determined as (2.6) through (2.5) based on the decomposition (2.3) and (2.4). We determine the critical value c_m so that

$$P\left(\frac{m}{n-p}F_{m, n-p} > c_m\right) = \alpha$$

and reject H_0 when $\bar{F} > c_m$. In this conditional test the critical value is determined by the number of positive components of $\pi_{\mathbf{S}}(\mathbf{Z}, \Theta)$.

Next, we consider the power of the test. Assume $\boldsymbol{\mu}^* \geq^* \mathbf{0}$ and consider the power of the alternative hypothesis $H_1 : \boldsymbol{\mu} = \boldsymbol{\mu}^*$. Since it is difficult to formulate the power, we obtain its approximate value for a specified $\boldsymbol{\Sigma}$ by Monte Carlo simulation. Specify the number of repetition of the experiment N . Let $R = 1$ and $R_1 = 0$. The algorithm is given as follows.

Step 1. Generate random numbers $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ from $N_p(\boldsymbol{\mu}^*, \boldsymbol{\Sigma})$.

Step 2. Determine $\pi_{\mathbf{S}}(\mathbf{Z}, \Theta)$ and $\bar{F}$ by $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$. If the number of positive components of $\pi_{\mathbf{S}}(\mathbf{Z}, \Theta)$ is m , determine the critical value c_m so that

$$P\left(\frac{m}{n-p} F_{m, n-p} > c_m\right) = \alpha.$$

Step 3.

$$\bar{F} > c_m \Rightarrow R_1 = R_1 + 1, \quad \bar{F} \leq c_m \Rightarrow R_1 = R_1.$$

Step 4. If $R \leq N$, go to Step 1. Otherwise, stop the experiment and let

$$P^* = \frac{R_1}{N}$$

be an approximate value of the power.

2.5 Simulation results

In this subsection we give some examples obtained by simulation. LRT denotes the test discussed in subsection 2.3. CLRT denotes the test discussed in subsection 2.4.

First, we give the actual Type I error intended to investigate the conservation of the critical value of LRT determined in subsection 2.3. Let $p = 3$. Let $\alpha = 0.05$ and $n = 20, 30$. By (2.9) we obtain conservative critical values of multivariate one-sided test $c = 0.503, 0.295$ for $n = 20, 30$, respectively. Table 1 gives Type I error for two tests by using conservative critical values for $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2$ where

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}, \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}.$$

Here ρ is set up so that $\boldsymbol{\Sigma}$ may be positive definite. Type I error in Table 1 is obtained by Monte Carlo simulation with $N = 100,000$. (All results by Monte Carlo simulation in this paper are calculated with $N = 100,000$.) Table shows that the critical value is less conservative as ρ is larger for $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2$. Type I error does not vary remarkably as the sample size n varies from 20 to 30.

Table 1: Type I error by conservative critical values of LRT

	ρ	-0.6	-0.3	0	0.3	0.6	0.9
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$	$n = 20$	—	0.0139	0.0215	0.0271	0.0342	0.0435
	$n = 30$	—	0.0140	0.0212	0.0270	0.0348	0.0421
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_2$	$n = 20$	0.0113	0.0157	0.0214	0.0254	0.0308	—
	$n = 30$	0.0107	0.0169	0.0206	0.0252	0.0312	—

Next, we give the power of the test for comparing LRT and CLRT. Let $\alpha = 0.05$ and $n = 20, 30$. If $n = 20$, the critical values of CLRT are

$$c_3 = 0.565, \quad c_2 = 0.423, \quad c_1 = 0.262$$

which are determined corresponding to the number of positive components of the orthogonal projection. If $n = 30$, the critical values of CLRT are

$$c_3 = 0.329, \quad c_2 = 0.249, \quad c_1 = 0.156.$$

Tables 2 to 4 give the power of the alternative hypothesis $H_1 : \boldsymbol{\mu} = \boldsymbol{\mu}^*$ where $\boldsymbol{\mu}^* = (0.5, 0.5, 0.5)'$, $(0.5, 0.5, 0)'$, $(0.5, 0, 0)'$ for $\alpha = 0.05$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2$. They show that the variations of the power of two tests corresponding to the variation of ρ for each H_1 are similar. Under $H_1 : \boldsymbol{\mu} = (0.5, 0.5, 0.5)'$ the power decreases as ρ increases for $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2$. Under $H_1 : \boldsymbol{\mu} = (0.5, 0.5, 0)'$ for $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$, the power decreases as ρ increases from -0.3 to a certain value, then the power increases as ρ increases. On the other hand, under $H_1 : \boldsymbol{\mu} = (0.5, 0.5, 0)'$ for $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_2$ the power decreases as ρ increases from -0.6 to 0.6 . Under $H_1 : \boldsymbol{\mu} = (0.5, 0, 0)'$ for $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$, the power decreases as ρ increases from -0.3 to a certain value, then the power increases as ρ increases. Under $H_1 : \boldsymbol{\mu} = (0.5, 0, 0)'$ for $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_2$, the variation of the power corresponding to the variation of ρ is similar to that for $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$. Next, we discuss the difference between LRT and CLRT. Under $H_1 : \boldsymbol{\mu} = (0.5, 0.5, 0.5)'$ LRT is more powerful compared to CLRT for almost all ρ . Under $H_1 : \boldsymbol{\mu} = (0.5, 0.5, 0)'$ and $H_1 : \boldsymbol{\mu} = (0.5, 0, 0)'$ CLRT is more powerful compared to LRT for almost all ρ . For each test the power with $n = 30$ is uniformly higher than that with $n = 20$. Although Type I error increases as ρ increases, the aspect of variation of the power does not correspond to that of Type I error.

Table 2: Power of $H_1 : \boldsymbol{\mu} = (0.5, 0.5, 0.5)'$

			ρ	-0.6	-0.3	0	0.3	0.6	0.9
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$	$n = 20$	LRT		—	0.999	0.875	0.680	0.541	0.442
		CLRT		—	0.998	0.839	0.622	0.475	0.382
	$n = 30$	LRT		—	1.000	0.980	0.881	0.755	0.646
		CLRT		—	1.000	0.971	0.846	0.700	0.588
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_2$	$n = 20$	LRT		1.000	0.985	0.872	0.751	0.741	—
		CLRT		1.000	0.976	0.842	0.699	0.687	—
	$n = 30$	LRT		1.000	1.000	0.980	0.927	0.919	—
		CLRT		1.000	0.999	0.972	0.901	0.893	—

Table 3: Power of $H_1 : \boldsymbol{\mu} = (0.5, 0.5, 0)'$

			ρ	−0.6	−0.3	0	0.3	0.6	0.9
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$	$n = 20$	LRT		—	0.918	0.666	0.614	0.729	0.997
		CLRT		—	0.925	0.677	0.625	0.740	0.998
	$n = 30$	LRT		—	0.992	0.878	0.837	0.921	1.000
		CLRT		—	0.992	0.884	0.846	0.926	1.000
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_2$	$n = 20$	LRT		0.991	0.850	0.666	0.563	0.557	—
		CLRT		0.990	0.855	0.679	0.577	0.568	—
	$n = 30$	LRT		0.998	0.974	0.879	0.792	0.784	—
		CLRT		0.998	0.977	0.886	0.801	0.793	—

Table 4: Power of $H_1 : \boldsymbol{\mu} = (0.5, 0, 0)'$

			ρ	−0.6	−0.3	0	0.3	0.6	0.9
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$	$n = 20$	LRT		—	0.326	0.269	0.335	0.526	0.924
		CLRT		—	0.361	0.295	0.356	0.548	0.925
	$n = 30$	LRT		—	0.503	0.458	0.506	0.716	0.929
		CLRT		—	0.518	0.447	0.530	0.731	0.928
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_2$	$n = 20$	LRT		0.460	0.274	0.270	0.321	0.622	—
		CLRT		0.490	0.306	0.296	0.344	0.636	—
	$n = 30$	LRT		0.590	0.423	0.420	0.489	0.784	—
		CLRT		0.608	0.453	0.449	0.514	0.792	—

3 Multivariate one-sided test for comparing two normal mean vectors

Next, we consider a multivariate one-sided test for comparing two normal mean vectors based on the likelihood ratio test.

3.1 Likelihood ratio test

We assume that there are two independent p -dimensional random variable vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ distributed according to multivariate normal $N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ and $N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$, respectively. We consider the test

$$H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 \text{ vs. } H_1 : \boldsymbol{\mu}_1 \geq^* \boldsymbol{\mu}_2 \quad (3.1)$$

based on the likelihood ratio test. Here $\boldsymbol{\mu}_1 \geq^* \boldsymbol{\mu}_2$ means $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 \geq^* \mathbf{0}$. Let $\mathbf{X}_{11}, \mathbf{X}_{12}, \dots, \mathbf{X}_{1n_1}$ and $\mathbf{X}_{21}, \mathbf{X}_{22}, \dots, \mathbf{X}_{2n_2}$ be samples from $N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ and $N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$, respectively. Let

$$\bar{\mathbf{X}}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \mathbf{X}_{ij} \quad (i = 1, 2), \quad \mathbf{Z} = \sqrt{\frac{n_1 n_2}{n_1 + n_2}} (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2),$$

$$\mathbf{S} = \sum_{i=1}^{n_1} (\mathbf{X}_{1i} - \bar{\mathbf{X}}_1)(\mathbf{X}_{1i} - \bar{\mathbf{X}}_1)' + \sum_{i=1}^{n_2} (\mathbf{X}_{2i} - \bar{\mathbf{X}}_2)(\mathbf{X}_{2i} - \bar{\mathbf{X}}_2)'.$$

By the likelihood ratio test criteria for (3.1) we reject H_0 when

$$\bar{F} = \frac{\|\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})\|_{\mathbf{S}}^2}{1 + \|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})\|_{\mathbf{S}}^2} > c$$

for a specified critical value c .

3.2 Null distribution of the likelihood ratio test statistic

Assume $\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})$ is determined as

$$\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}) = \begin{pmatrix} \mathbf{Z}_{1 \cdot 2} \\ \mathbf{0} \end{pmatrix} \quad (3.2)$$

through

$$\mathbf{Z}_{1 \cdot 2} = \mathbf{Z}_1 - \mathbf{S}_{12} \mathbf{S}_{22}^{-1} \mathbf{Z}_2 > \mathbf{0}, \quad \mathbf{S}_{22}^{-1} \mathbf{Z}_2 \leq \mathbf{0} \quad (3.3)$$

based on the decompositions

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix}$$

where the dimension of $\mathbf{Z}_1$ is m and $\mathbf{S}_{11}$ is the $m \times m$ matrix. Then

$$\bar{F} = \frac{\mathbf{Z}_{1 \cdot 2}' \mathbf{S}_{11 \cdot 2} \mathbf{Z}_{1 \cdot 2}}{1 + \mathbf{Z}_2' \mathbf{S}_{22}^{-1} \mathbf{Z}_2} \quad (3.4)$$

where $\mathbf{S}_{11 \cdot 2} = \mathbf{S}_{11} - \mathbf{S}_{12} \mathbf{S}_{22}^{-1} \mathbf{S}_{21}$. Since $\mathbf{S} \sim W_p(n_1 + n_2 - 2, \boldsymbol{\Sigma})$, we obtain

$$\mathbf{S}_{11 \cdot 2} \sim W_m(n_1 + n_2 - 2 - p + m, \boldsymbol{\Sigma}_{11 \cdot 2}) \quad (3.5)$$

by Perlman (1969) [4] 3.2° (P. 551). Here $\boldsymbol{\Sigma}_{11 \cdot 2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21}$. Furthermore, by Perlman (1969) [4] 3.7° (P. 551)

$$\frac{\mathbf{Z}_{1 \cdot 2}}{\sqrt{1 + \mathbf{Z}_2' \mathbf{S}_{22}^{-1} \mathbf{Z}_2}} \sim N_m(\mathbf{0}, \boldsymbol{\Sigma}_{11 \cdot 2}) \quad (3.6)$$

under H_0 . $\mathbf{Z}$ and $\mathbf{S}$ are independent. Furthermore, $\mathbf{S}_{11 \cdot 2}$ is distributed independently of $\mathbf{S}_{12}$ and $\mathbf{S}_{22}$ by Perlman (1969) [4] 3.2° (P. 551). Therefore, $\mathbf{S}_{11 \cdot 2}$ and $\mathbf{Z}_{1 \cdot 2} / \sqrt{1 + \mathbf{Z}_2' \mathbf{S}_{22}^{-1} \mathbf{Z}_2}$ are independent. By (3.4), (3.5) and (3.6)

$$\bar{F} \sim \frac{\chi_m^2}{\chi_{n_1 + n_2 - 1 - p}^2} \sim \frac{m}{n_1 + n_2 - 1 - p} F_{m, n_1 + n_2 - 1 - p}. \quad (3.7)$$

The distribution (3.7) is independent of the event (3.3). Letting $P(m)$ be the probability that the number of positive components of $\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})$ is m ,

$$P(\bar{F} > c) = \sum_{m=1}^p P(m) P\left(\frac{m}{n_1 + n_2 - 1 - p} F_{m, n_1 + n_2 - 1 - p} > c\right).$$

However, $P(m)$ depends on unknown $\boldsymbol{\Sigma}$.

3.3 Determination of conservative critical value

Since the distribution of $\bar{F}$ depends on the unknown $\boldsymbol{\Sigma}$ under H_0 , we can not determine the critical value c satisfying

$$P(\bar{F} > c) = \alpha$$

for a specified significance level α independently of Σ . However, we derive the upper bound of the distribution of $\bar{F}$ for all covariance matrices under H_0 referring to Perlman (1969) [4]. Therefore, we can obtain a conservative critical value using it. The bound is given by

$$\begin{aligned} \max_{\Sigma} P(\bar{F} > c) &= \frac{1}{2}P(((p-1)/(n_1+n_2-1-p))F_{p-1, n_1+n_2-1-p} > c) \\ &+ \frac{1}{2}P((p/(n_1+n_2-1-p))F_{p, n_1+n_2-1-p} > c). \end{aligned}$$

If we determine c so that

$$\begin{aligned} &\frac{1}{2}P(((p-1)/(n_1+n_2-1-p))F_{p-1, n_1+n_2-1-p} > c) \\ &+ \frac{1}{2}P((p/(n_1+n_2-1-p))F_{p, n_1+n_2-1-p} > c) = \alpha, \end{aligned} \tag{3.8}$$

we obtain

$$P(\bar{F} > c) \leq \alpha,$$

which shows c is a conservative critical value.

The actual Type I error is given by $P(\bar{F} > c)$ under H_0 . We obtain its approximate value for a specified Σ by Monte Carlo simulation. Specify the number of repetition of the experiment N . Assume $\mu_1 = \mu_2 = \mathbf{0}$ without loss of generality. Let $R = 1$ and $R_1 = 0$. The algorithm is given as follows.

Step 1. Generate random numbers $\mathbf{X}_{11}, \mathbf{X}_{12}, \dots, \mathbf{X}_{1n_1}$ and $\mathbf{X}_{21}, \mathbf{X}_{22}, \dots, \mathbf{X}_{2n_2}$ from $N_p(\mathbf{0}, \Sigma)$ and $N_p(\mathbf{0}, \Sigma)$, respectively.

Step 2. Determine $\bar{F}$ by $\mathbf{X}_{11}, \mathbf{X}_{12}, \dots, \mathbf{X}_{1n_1}$ and $\mathbf{X}_{21}, \mathbf{X}_{22}, \dots, \mathbf{X}_{2n_2}$. Then

$$\bar{F} > c \Rightarrow R_1 = R_1 + 1, \quad \bar{F} \leq c \Rightarrow R_1 = R_1.$$

Step 3. $R = R + 1$. If $R \leq N$, go to Step 1. Otherwise, stop the experiment and let

$$P^* = \frac{R_1}{N}$$

be an approximate value of $P(\bar{F} > c)$.

Next, we consider the power of the test. Assuming that p -dimensional vector μ^* satisfies $\mu^* \geq^* \mathbf{0}$, we consider the power of the alternative hypothesis $H_1 : \mu_1 - \mu_2 = \mu^*$. The power is $P(\bar{F} > c)$ under H_1 . We calculate its approximate value for a specified Σ by Monte Carlo simulation. Assume $\mu_1 = \mu^*, \mu_2 = \mathbf{0}$ without loss of generality. Specify the number of repetition of the experiment N in advance. Let $R = 1$ and $R_1 = 0$. The algorithm is given as follows.

Step 1. Generate random numbers $\mathbf{X}_{11}, \mathbf{X}_{12}, \dots, \mathbf{X}_{1n_1}$ and $\mathbf{X}_{21}, \mathbf{X}_{22}, \dots, \mathbf{X}_{2n_2}$ from $N_p(\mu^*, \Sigma)$ and $N_p(\mathbf{0}, \Sigma)$, respectively.

Step 2. Determine $\bar{F}$ by using $\mathbf{X}_{11}, \mathbf{X}_{12}, \dots, \mathbf{X}_{1n_1}$ and $\mathbf{X}_{21}, \mathbf{X}_{22}, \dots, \mathbf{X}_{2n_2}$. Then

$$\bar{F} > c \Rightarrow R_1 = R_1 + 1, \quad \bar{F} \leq c \Rightarrow R_1 = R_1.$$

Step 3. $R = R + 1$. If $R \leq N$, go to Step 1. Otherwise, stop the experiment and let

$$P^* = \frac{R_1}{N}$$

be an approximate value of $P(\bar{F} > c)$ under H_1 .

3.4 Conditional likelihood ratio test

We propose a conditional test for (3.1) based on the likelihood ratio test referring to Fraser *et al.* (1991) [1]. Specifically, we test (3.1) under the condition that the orthogonal projection $\pi_{\mathcal{S}}(\mathbf{Z}, \Theta)$ is determined as (3.2) through (3.3). We determine the critical value c_m so that

$$P\left(\frac{m}{n_1 + n_2 - 1 - p} F_{m, n_1 + n_2 - 1 - p} > c_m\right) = \alpha$$

and reject H_0 when $\bar{F} > c_m$. In this conditional test the critical value is determined by the number of positive components of $\pi_{\mathcal{S}}(\mathbf{Z}, \Theta)$.

Next, we consider the power of the test. Assuming that p -dimensional vector $\boldsymbol{\mu}^*$ satisfies $\boldsymbol{\mu}^* \geq^* \mathbf{0}$, we consider the power of the alternative hypothesis $H_1 : \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \boldsymbol{\mu}^*$. We obtain its approximate value for a specified $\boldsymbol{\Sigma}$ by Monte Carlo simulation. Assume $\boldsymbol{\mu}_1 = \boldsymbol{\mu}^*, \boldsymbol{\mu}_2 = \mathbf{0}$ without loss of generality. Specify the number of repetition of the experiment N . Let $R = 1$ and $R_1 = 0$. The algorithm is given as follows.

Step 1. Generate random numbers $\mathbf{X}_{11}, \mathbf{X}_{12}, \dots, \mathbf{X}_{1n_1}$ and $\mathbf{X}_{21}, \mathbf{X}_{22}, \dots, \mathbf{X}_{2n_2}$ from $N_p(\boldsymbol{\mu}^*, \boldsymbol{\Sigma})$ and $N_p(\mathbf{0}, \boldsymbol{\Sigma})$, respectively.

Step 2. Determine $\pi_{\mathcal{S}}(\mathbf{Z}, \Theta)$ and $\bar{F}$ by $\mathbf{X}_{11}, \mathbf{X}_{12}, \dots, \mathbf{X}_{1n_1}$ and $\mathbf{X}_{21}, \mathbf{X}_{22}, \dots, \mathbf{X}_{2n_2}$. If the number of positive components of $\pi_{\mathcal{S}}(\mathbf{Z}, \Theta)$ is m , determine the critical value c_m so that

$$P\left(\frac{m}{n_1 + n_2 - 1 - p} F_{m, n_1 + n_2 - 1 - p} > c_m\right) = \alpha.$$

Step 3.

$$\bar{F} > c_m \Rightarrow R_1 = R_1 + 1, \quad \bar{F} \leq c_m \Rightarrow R_1 = R_1.$$

Step 4. If $R \leq N$, go to Step 1. Otherwise, stop the experiment and let

$$P^* = \frac{R_1}{N}$$

be an approximate value of the power.

3.5 Simulation results

In this subsection we give some examples obtained by simulation. LRT denotes the test discussed in subsection 3.3. CLRT denotes the test discussed in subsection 3.4.

First, we give the actual Type I error intended to investigate the conservation of the critical value of LRT determined in subsection 3.3. Let $p = 3$. Let $\alpha = 0.05$ and $n_1 + n_2 = 50$. By (3.8) we obtain conservative critical value of multivariate one-sided test $c = 0.165$. Table 5 gives Type I error by using the conservative critical value for $(n_1, n_2) = (25, 25), (30, 20), (40, 10)$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2$ where

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}, \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}.$$

Table shows that the critical value is less conservative as ρ is larger for $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2$. There do not exist remarkable differences of Type I error among three types of pairs of sample sizes.

Table 5: Type I error by conservative critical values of LRT

ρ		-0.6	-0.3	0	0.3	0.6	0.9
$\Sigma = \Sigma_1$	$(n_1, n_2) = (25, 25)$	—	0.0132	0.0206	0.0267	0.0326	0.0427
	$(n_1, n_2) = (30, 20)$	—	0.0130	0.0207	0.0277	0.0340	0.0418
	$(n_1, n_2) = (40, 10)$	—	0.0134	0.0217	0.0269	0.0337	0.0413
$\Sigma = \Sigma_2$	$(n_1, n_2) = (25, 25)$	0.0113	0.0164	0.0206	0.0251	0.0309	—
	$(n_1, n_2) = (30, 20)$	0.0117	0.0158	0.0211	0.0250	0.0305	—
	$(n_1, n_2) = (40, 10)$	0.0114	0.0164	0.0209	0.0259	0.0307	—

Next, we give the power of the test for two tests. Let $\alpha = 0.05$ and $n_1 + n_2 = 50$. The critical values of CLRT are

$$c_3 = 0.184, c_2 = 0.140, c_1 = 0.089$$

which are determined corresponding to the number of positive components of the orthogonal projection. Tables 6 to 8 give the power for two tests under the alternative hypothesis $H_1 : \mu_1 - \mu_2 = \mu^*$ where $\mu^* = (0.5, 0.5, 0.5)'$, $(0.5, 0.5, 0)'$, $(0.5, 0, 0)'$. They show that the variations of the power of two tests corresponding to the variation of ρ for each H_1 are similar. Specifically, they are similar to those given in subsection 2.5. Next, we discuss the differences of the power corresponding to the type of pair of sample sizes. The power by (25, 25) is highest compared to that by (30, 20), (40, 10) in each case. The power by (30, 20) is higher compared to that by (40, 10) in each case. Next, we discuss the difference between LRT and CLRT. Under $H_1 : \mu_1 - \mu_2 = (0.5, 0.5, 0.5)'$ LRT is more powerful compared to CLRT for almost all ρ . Under $H_1 : \mu_1 - \mu_2 = (0.5, 0.5, 0)'$ and $H_1 : \mu_1 - \mu_2 = (0.5, 0, 0)'$ CLRT is more powerful compared to LRT for almost all ρ . These results are similar to those in subsection 2.5.

Table 6: Power of $H_1 : \mu = (0.5, 0.5, 0.5)'$

ρ			-0.6	-0.3	0	0.3	0.6	0.9
$\Sigma = \Sigma_1$	$(n_1, n_2) = (25, 25)$	LR	—	0.987	0.738	0.529	0.405	0.332
		CLR	—	0.984	0.700	0.477	0.357	0.279
	$(n_1, n_2) = (30, 20)$	LR	—	0.984	0.719	0.511	0.391	0.323
		CLR	—	0.981	0.679	0.458	0.343	0.269
	$(n_1, n_2) = (40, 10)$	LR	—	0.908	0.535	0.362	0.278	0.231
		CLR	—	0.904	0.498	0.324	0.241	0.193
$\Sigma = \Sigma_2$	$(n_1, n_2) = (25, 25)$	LR	0.999	0.930	0.740	0.595	0.586	—
		CLR	0.999	0.916	0.700	0.549	0.537	—
	$(n_1, n_2) = (30, 20)$	LR	0.998	0.920	0.716	0.579	0.568	—
		CLR	0.998	0.905	0.683	0.532	0.520	—
	$(n_1, n_2) = (40, 10)$	LR	0.993	0.770	0.534	0.411	0.405	—
		CLR	0.995	0.752	0.499	0.374	0.366	—

Table 7: Power of $H_1 : \boldsymbol{\mu} = (0.5, 0.5, 0)'$

ρ			−0.6	−0.3	0	0.3	0.6	0.9
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$	$(n_1, n_2) = (25, 25)$	LR	—	0.797	0.508	0.464	0.573	0.981
		CLR	—	0.808	0.521	0.477	0.585	0.982
	$(n_1, n_2) = (30, 20)$	LR	—	0.781	0.489	0.449	0.553	0.977
		CLR	—	0.792	0.501	0.459	0.564	0.977
	$(n_1, n_2) = (40, 10)$	LR	—	0.581	0.334	0.305	0.386	0.885
		CLR	—	0.601	0.347	0.318	0.395	0.890
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_2$	$(n_1, n_2) = (25, 25)$	LR	0.969	0.699	0.509	0.416	0.415	—
		CLR	0.971	0.713	0.523	0.430	0.427	—
	$(n_1, n_2) = (30, 20)$	LR	0.967	0.681	0.489	0.403	0.401	—
		CLR	0.966	0.691	0.503	0.415	0.413	—
	$(n_1, n_2) = (40, 10)$	LR	0.908	0.483	0.333	0.277	0.273	—
		CLR	0.912	0.501	0.347	0.286	0.283	—

Table 8: Power of $H_1 : \boldsymbol{\mu} = (0.5, 0, 0)'$

ρ			−0.6	−0.3	0	0.3	0.6	0.9
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$	$(n_1, n_2) = (25, 25)$	LR	—	0.241	0.196	0.245	0.400	0.903
		CLR	—	0.270	0.219	0.267	0.422	0.908
	$(n_1, n_2) = (30, 20)$	LR	—	0.230	0.188	0.236	0.384	0.901
		CLR	—	0.262	0.210	0.255	0.406	0.903
	$(n_1, n_2) = (40, 10)$	LR	—	0.155	0.131	0.160	0.264	0.802
		CLR	—	0.180	0.148	0.178	0.279	0.812
$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_2$	$(n_1, n_2) = (25, 25)$	LR	0.354	0.199	0.197	0.234	0.485	—
		CLR	0.390	0.228	0.218	0.256	0.507	—
	$(n_1, n_2) = (30, 20)$	LR	0.344	0.192	0.188	0.225	0.472	—
		CLR	0.379	0.217	0.209	0.246	0.493	—
	$(n_1, n_2) = (40, 10)$	LR	0.233	0.131	0.129	0.154	0.328	—
		CLR	0.267	0.152	0.149	0.171	0.350	—

4 Example

We give an example made by artificial data in order to illustrate our procedures. A similar example is discussed in Imada and Douke (2007) [2]. We have two treatments A_1, A_2 . Here A_1 is a new treatment and A_2 is a standard treatment. Each treatment is evaluated by three measurements X_1, X_2, X_3 . For instance, in the example given by Tang *et al.* (1989) [5] a treatment is evaluated by efficacy and two toxicities. Letting $\mathbf{X}_1, \mathbf{X}_2$ be response vectors of A_1, A_2 , assume

$$\mathbf{X}_1 \sim N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}), \quad \mathbf{X}_2 \sim N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}).$$

25 patients are treated with each treatment. Means of observations are given by Table 9.

Table 9: Sample means for each measurement

	X_1	X_2	X_3
A_1	0.277	0.405	-0.108
A_2	-0.050	-0.170	0.236

Furthermore, we obtain

$$\mathbf{S} = \begin{pmatrix} 39.159 & 7.087 & 10.704 \\ 7.087 & 41.148 & 6.986 \\ 10.704 & 6.986 & 42.737 \end{pmatrix}.$$

We test

$$H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 \text{ vs. } H_1 : \boldsymbol{\mu}_1 \geq^* \boldsymbol{\mu}_2$$

for $\alpha = 0.05$. Here, H_1 means that A_1 is superior to A_2 . Then

$$\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}) = (1.461, 2.229, 0)', \quad \bar{F} = 0.157.$$

Since $\bar{F} < c = 0.165$, we do not reject H_0 by LRT. However, since $\bar{F} > c_2 = 0.140$, we reject H_0 by CLRT. Although we do not decide that A_1 is superior to A_2 by LRT, we decide that A_1 is superior to A_2 by CLRT.

5 Conclusions

In this study, we have discussed a multivariate one-sided test for comparing two normal mean vectors based on the likelihood ratio test under the assumption that a common covariance matrix is unknown. Specifically, assuming that each component of one vector is not smaller than that of the other vector, we proposed methods for testing the difference between two normal mean vectors. Since it is difficult to derive the distribution of the likelihood ratio test statistic, we proposed two methods to carry out the test for a specified significance level. One was a test by using a conservative critical value constructed by the upper bound of the null distribution of the likelihood ratio test statistic. The other was a test under the condition that the orthogonal projection of the observed point onto the positive orthant is determined. We gave some examples regarding critical values and the power of the test obtained by Monte Carlo simulation intended to compare two tests. Furthermore, we gave an example in order to illustrate our procedures. Our multivariate one-sided test can be applied to comparing two treatments evaluated by multivariate normal response in clinical trials. Specifically, it is possible to test the superiority of a treatment compared to another treatment. However, we occasionally want to test the superiority and the inferiority between two treatments. Therefore, assuming that all differences of components between two normal mean vectors are simultaneously nonnegative or nonpositive, we should discuss a multivariate two-sided test for testing whether two normal mean vectors are different in the future.

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