Proc. Sch. Sci. TOKAI UNIV.
49 (2014) 1–21

On the Theory of Gordan-Noether on Homogeneous Forms with Zero Hessian

by

Junzo Watanabe

Department of Mathematics
School of Science, Tokai University
4-1-1 Kitakaname, Hiratsuka-shi, Kanagawa, 259-1292, Japan
E-mail:watanabe.junzo@tokai-u.jp

(Received on June 2, 2013; accepted on June 14, 2013)

I dedicate this paper to the memory of Professor Hiroshi Yamada (1933-2010). He was one of my teachers at Nagoya University and he introduced me to Gordan-Noether's paper. It was a source of inspiration through my life.

Abstract

We give a detailed proof for Gordan-Noether's results in "Ueber die algebraischen Formen, deren Hesse'sche Determinante identisch verschwindet." C. Lossen has written a paper in a similar direction as the present paper. However, our paper is independent of Lossen's paper and includes a considerable number of new observations.

Contents

1 Introduction **1**
2 Notation and preliminaries **4**
3 Self-vanishing systems of polynomials **5**
4 Forms with zero Hessian and reduced self-vanishing systems **10**
5 Binary and ternary forms with zero Hessian **13**
6 The rational map defined by h **13**
7 Quaternary and quinary forms with zero Hessian **18**

1 Introduction

In 1852 and 1856, O. Hesse wrote two papers in Crelle's Journal Bd. 42 and Bd. 56 in which he claimed that if the Hessian determinant of a homogeneous polynomial identically vanishes, then a variable can be eliminated by a linear transformation of the variables. Unfortunately his claim

is not true in general. In fact Hesse's proof was unconventional and the validity of the proof was questioned from the beginning [6]. Nonetheless it should have been easy to see that Hesse's claim is true for binary forms as well as quadrics. In 1875, M. Pasch proved that Hesse's claim is true for ternary cubics and quaternary cubics [10].

In 1876, P. Gordan and M. Noether [6] finally established the correct statement which says that if a form has zero Hessian, then one variable can be eliminated by a birational transformation of the variables. Moreover they proved in the same paper that Hesse's claim is true if the number of variables is at most four, and furthermore they determined all homogeneous polynomials in five variables for which the Hessian determinant identically vanishes.

The present paper goes beyond the necessity and desire to understand their proof. The Hessian of a homogeneous polynomial is essential to the theory of Artinian Gorenstein rings because it is used with higher Hessians to determine the set of the strong Lefschetz elements in a zero-dimensional Gorenstein algebra ([9]). In particular, if the Hessian of a homogeneous polynomial is identically zero, we get a Gorenstein algebra which lacks the strong Lefschetz property. To explain this further, let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K of characteristic zero and let $G \in R$ be a homogeneous polynomial. In addition let $I \subset R$ be the ideal:

$$I = \left\{ f(x_1, \ldots, x_n) \in R \,\middle|\, f\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) G = 0 \right\},$$

and let $A = R/I$. Then A is a zero-dimensional Gorenstein graded algebra $A = \bigoplus_{i=0}^{d} A_i$, where $d = \deg G$. It is easy to see that if the polynomial G contains properly n variables, then the partial derivatives $G_1, \ldots, G_n$ of G are linearly independent. Moreover if $L = \xi_1 x_1 + \cdots + \xi_n x_n \in R_1$ is a linear form, it defines a linear map

$$\times L^{d-2} : A_1 \longrightarrow A_{d-1}$$

by $A_1 \ni a \mapsto aL^{d-2} \in A_{d-1}$. Let M be the matrix for this linear map with respect to the bases $\langle x_1, \ldots, x_n \rangle$ and $\langle G_1, \ldots, G_n \rangle$ for A_1 and A_{d-1} respectively. Then $\det M$ is the Hessian of G evaluated at $(\xi_1, \ldots, \xi_n)$ (up to a constant multiple). If $\dim A_1 \leq 4$, and if we take the results of [6] for granted, this is to say that there exists a linear form L such that $L^{d-2} \neq 0$. If $n = 5$, Gordan-Noether's paper enables us to determind all homogeneous polynomials G such that L^{d-2} is zero for any choice of $\xi_1, \ldots, \xi_n$.

Gordan-Noether's paper [6] has been cited by several authors ([7], [9], [12]); however each time it had to be accompanied with a proviso that the result is yet to be confirmed. Gordan-Noether's paper is difficult to understand. Not only their results but also the methods have been completely forgotten. Thus, it seems necessary to consider the paper from the viewpoint of contemporary algebra. H. Yamada [13] devoted considerable efforts to constructing a modernized translation of [6]; however, it was not completely successful and therefore it was unpublished.

The purpose of this paper is to give detailed proofs for most of the results that were obtained by Gordan-Noether in [6]. The foundation of their theory lies in the fact that if the Hessian determinant of a homogeneous polynomial is identically zero, then the polynomial satisfies a certain linear partial differential equation. For simplicity we assume that the Hessian matrix $M := (\partial^2 f / \partial x_i \partial x_j)$ has corank one. Then the null space of the matrix M has dimension one over the function field $K(x)$. Let $(h_1, \ldots, h_n)$ be a vector with $h_j \in K(x)$ such that $(h_1, \ldots, h_n)M = 0$.

Then it is easy to see that f satisfies the partial differential equation

$$h_1\frac{\partial F}{\partial x_1}+h_2\frac{\partial F}{\partial x_2}+\cdots+h_n\frac{\partial F}{\partial x_n}=0 \tag{1}$$

By clearing the denominator we may think h_j are homogeneous polynomials of the same degree. (Cf. Remark 4.6.) Gordan and Noether discovered that each coefficient h_j itself of (1) satisfies the partial differential equation (1). This readily proves that a variable can be eliminated by a birational transformation. (See Theorem 4.7.) The Gordan-Noether called the functions satisfying (1) "die Functionen Φ." The coefficients of this partial linear differential equation were termed as a "self-vanishing system" by Yamada [13]. The solution of this type of differential equation behaves as if the coefficients were constants. According to [6], Jacobi considered this type of differential equation and it is the key to understanding the Gordan and Noether theory.

To prove that Hesse's claim is true for homogeneous polynomials for $n \leq 4$, it is necessary to consider the fundamental locus and the image of the rational map $\mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ defined by $(x_1, \ldots, x_n) \mapsto (h_1, \ldots, h_n)$. From the fact that $h_1, \ldots, h_n$ are a "self-vanishing system," it follows that the dimension of the image is significantly small. This enables us to determine the forms with zero Hessian for $n \leq 5$.

Gordan and Noether's idea and proof techniques appear quite new and they are interesting in their own right, and they give us a series of new problems, some of which will be investigated in our subsequent papers.

Historical notes

In 1990, H. Yamada wrote a paper[13], under Grant-in-Aid no.20022551 entitled "On the hypersurface defined by a form whose Hessian identically vanishes." It was unpublished because it was incomplete; however, he defined and systematically studied "the self-vanishing system of polynomials," as he named them. This paper was written to finish Yamada's paper [13].

This paper was written without a prior knowledge of Lossen's paper [8] and I gave a 3 hour lecture on this subject at the workshop "Aspects of SLP and WLP" held in Hawaii Tokai International College in Honolulu in September 2012, where I learned that Lossen had written a paper [8] in the same direction and that there were other related papers [2] and [3]. Clearly the objectives of Lossen's paper and this paper are identical. However the methods employed are different, although both are based on the same source, i.e., on the paper of Gordan and Noether [6].

The main differences are listed below.

1. Theorem 4.7 is not explicitly written in Lossen [8], which tells us that Hesse was nearly correct in his intuition, when he said that if the Hessian determinant vanishes, then a variable can be eliminated by linear transformation.
2. In this paper a "self-vanishing system" is defined and studied systematically.
3. Our observations Lemma 5.2 and Proposition 6.2 conciderably simplify the entire argument. I owe the proof of Proposition 6.2 to T. Maeno.
4. In Lossen's paper [8], the connection to the Lefschetz properties of Artinian rings is not indicated.

5. In this paper it is proved that a cubic form in five variables is essentially unique, while in [8] it is not necessarily clear.

I would like to thank H. Nasu and T. Tsukioka for insightful discussions for Remark 7.5.

2 Notation and preliminaries

Throughout the paper we denote by K an algebraically closed field of characteristic zero. We denote by $K[x_1, x_2, \ldots, x_n]$ the polynomial ring in the variables $x_1, \ldots, x_n$ and by $K(x_1, \ldots, x_n)$ the function field. Let $R = K[x_1, \ldots, x_n]$. An element of R is sometimes abbreviated as $f(x)$ or simply as f.

A **system** of homogeneous polynomials (or forms) of R is a vector $(f_1, f_2, \ldots, f_n)$ consisting of homogeneous polynomials $f_i \in R$ of the same degree. A system of forms is denoted by a bold face letter as $\boldsymbol{f} = (f_1, \ldots, f_n)$. To avoid triviality we will always assume that $\boldsymbol{f} \neq 0$. To indicate that a system is a vector of polynomials depending on the argument vector $x = (x_1, \ldots, x_n)$, we write $\boldsymbol{f}(x)$ as well as $\boldsymbol{f}$. Although a coordinate system like $x = (x_1, \ldots, x_n)$ is a "system of forms," we do not apply the rule to use a bold face letter to denote it. If $y = (y_1, y_2, \ldots, y_n)$ is a new set of variables and if $\boldsymbol{f}(x)$ is a system of forms in R, then $\boldsymbol{f}(y)$ obtained from $\boldsymbol{f}(x)$ by the substitution $x_j \to y_j$ is a system of forms in $K[y_1, \ldots, y_n]$.

We treat vectors both as row vectors and as column vectors, so if $A = (a_{ij})$ is an $n \times n$ matrix with $a_{ij} \in K$, and if $x = (x_1, \ldots, x_n)$ is a vector, then $y = Ax$ means that $y = (y_1, \ldots, y_n)$ is a vector defined by

$$y_i = \sum_{j=1}^{n} a_{ij} x_j.$$

Likewise $y = xA$ means that

$$y_j = \sum_{i=1}^{n} a_{ij} x_i.$$

The same rule applies to systems of polynomials as well as coordinate systems. Thus, if $\boldsymbol{f} = (f_1, \ldots, f_n)$ is a system of forms, then $\boldsymbol{f}' = A\boldsymbol{f}$ is a system of forms for any $n \times n$ invertible matrix A over K.

Lemma 2.1. *Let $K[x] = K[x_1, \ldots, x_n]$ be the polynomial ring and let $f(x) = f(x_1, \ldots, x_n) \in K[x]$. Let $A = (a_{ij}) \in \mathrm{GL}(n, K)$, and put*

$$(x'_1, x'_2, \ldots, x'_n) = A(x_1, x_2, \ldots, x_n)$$

Consider $(x'_1 \ldots, x'_n)$ as a new coordinate system and let f' be the polynomial in $x'_1, \ldots, x'_n$ defined by

$$f'(x') := f(A^{-1}x') = f(x).$$

Then

$$(f'_1, f'_2, \ldots, f'_n) = {}^tA^{-1}(f_1, f_2, \ldots, f_n)$$

where $f_j = \frac{\partial f}{\partial x_j}$, and $f'_j = \frac{\partial f'}{\partial x'_j}$.

Proof is left to the reader.

Lemma 2.2. *Let $f(x) = f(x_1, \dots, x_n)$ be a homogeneous polynomial of positive degree. Put $f_i = \frac{\partial f}{\partial x_i}$ and let $\boldsymbol{f} = (f_1, \dots, f_n)$ be a system of polynomials. If $\dim_K \sum_{i=1}^n K f_i = s$, then $n - s$ variables can be eliminated from f by means of a linear transformation of the variables. In other words, there exists an invertible matrix $A = (a_{ij}) \in \mathrm{GL}(n, K)$ such that, if we let $x' = Ax$, then the polynomial $f'(x'_1, \dots, x'_n) = f(A^{-1}x')$ does not depend on $x'_{s+1}, x'_{s+2}, \dots, x'_n$.*

Proof. By assumption there exists an invertible matrix B such that $B(f_1, f_2, \dots, f_n) = (f'_1, f'_2, \dots, f'_s, 0, 0, \dots, 0)$. Let A be the matrix such that ${}^tA^{-1} = B$, and put $x' = Ax$ and $f'(x') = f(A^{-1}x')$. Then

$$\frac{\partial f'}{\partial x'_j} = 0 \quad \text{for} \quad j \geq s+1,$$

by Lemma 2.1. Thus $f'(x')$ does not depend on the variables $x'_{s+1}, x'_{s+2}, \dots, x'_n$. □

Proposition 2.3. *Let $\boldsymbol{f} = (f_1, \dots, f_n)$ be a system of forms in $K[x_1, \dots, x_n]$, where $f_1, \dots, f_n$ are algebraically dependent. Let $y = (y_1, \dots, y_n)$ be a new set of variables independent of x. Let*

$$\phi : K[y_1, y_2, \dots, y_n] \to K[x_1, x_2, \dots, x_n]$$

be the homomorphism defined by $y_i \mapsto f_i$, and let $g = g(y_1, \dots, y_n)$ be a nonzero element in $\ker \phi$ of the least degree. Put

$$h_j = h_j(x) := \frac{\partial g}{\partial y_j}(f_1, \dots, f_n).$$

Namely, $h_j(x)$ is the polynomial obtained from $\frac{\partial g}{\partial y_j}$ by substituting $(y_1, \dots, y_n)$ for $(f_1, \dots, f_n)$. Let $W = \sum_{j=1}^n K h_j$ be the vector space over K spanned by the elements $h_1, \dots, h_n$. Let $s = \dim_K W$. Then $n - s$ variables can be eliminated from $g(y)$ by means of a linear transformation of the variables $y_1, \dots, y_n$.

Proof. Put $V = \sum_{i=1}^n K \frac{\partial g}{\partial y_i}$. We claim that $\dim_K V = s$. It is clear that $\dim_K V \geq s$. Consider the restriction

$$\phi|_V : V \to K[x_1, x_2, \dots, x_n].$$

We see that $(\ker \phi) \cap V = 0$ by minimality of the degree of g. Thus we have $\dim_K V = s$, since $\phi|_V$ is injective and $\operatorname{im} \phi$ is W. By the previous lemma, proof is complete. □

Remark 2.4. In the above Proposition, it is possible that $h_j = 0$ for some j. It means that $\frac{\partial g}{\partial y_j} = 0$. Hence $\boldsymbol{h} := (h_1, \dots, h_n) \neq 0$. If we drop the condition that $\deg g$ is minimal in $\mathcal{I}$, we may define $\boldsymbol{h}$ as well, but $\boldsymbol{h}$ can be 0.

3 Self-vanishing systems of polynomials

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be two sets of indeterminates and let $K(x, y) = K(x_1, \dots, x_n, y_1, \dots, y_n)$ denote the rational function field. We introduce the differential operator

$$\mathcal{D}_x(y) : K(x, y) \to K(x, y),$$

which is defined by

$$\mathcal{D}_x(y)f(x,y) := \sum_{j=1}^{n} y_j \frac{\partial f(x,y)}{\partial x_j}$$

for $f(x,y) \in K(x,y)$. For a homogeneous polynomial $f(x) = f(x_1,\ldots,x_n) \in K[x_1,\ldots,x_n]$ and for $j \geq 0$, define $f^{(j)}(x,y)$ to be the polynomial in $K[y_1,\ldots,y_n,x_1,\ldots,x_n]$ given by

$$f^{(j)}(x,y) = \frac{1}{j!}\mathcal{D}_x(y)^j(f(x)).$$

It is easy to see that

$$f^{(j)}(x,x) = \binom{d}{j} f(x), \quad j = 0,1,2,\cdots,d.$$

and where $d = \deg f$. In particular

$$f^{(d)}(x,y) = f(y).$$

Proposition 3.1. *Let $A = (a_{ij})$ be an invertible matrix with $a_{ij} \in K$ and let*

$$x' = (x'_1,\ldots,x'_n) = A(x_1,\ldots,x_n),$$

$$y' = (y'_1,\ldots,y'_n) = A(y_1,\ldots,y_n).$$

Let $f'(x')$ be the polynomial in the coordinate x' defined by $f'(x') = f(x) \in K[x_1,\ldots,x_n]$. Then we have

$$\mathcal{D}_{x'}(y')f'(x') = \mathcal{D}_x(y)f(x).$$

Proof. Let $\boldsymbol{f}$ be the system of forms:

$$\boldsymbol{f} = \frac{\partial f(x)}{\partial x} := \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n}\right).$$

Likewise, let $\boldsymbol{f}'$ be the system of forms:

$$\boldsymbol{f}' = \frac{\partial f'(x')}{\partial x'} = \left(\frac{\partial f'(x')}{\partial x'_1}, \frac{\partial f'(x')}{\partial x'_2}, \ldots, \frac{\partial f'(x')}{\partial x'_n}\right).$$

Then, by definition,

$$\mathcal{D}_x(y)f(x) = y \cdot \boldsymbol{f} = y_1\frac{\partial f}{\partial x_1} + \cdots + y_n\frac{\partial f}{\partial x_n}.$$

By Lemma 2.1 we have

$$\begin{aligned}\mathcal{D}_x(y)f(x) = y \cdot \boldsymbol{f} &= \left(A^{-1}y'\right) \cdot \left({}^tA\boldsymbol{f}'\right) \\ &= \left(y'^tA^{-1}\right) \cdot \left({}^tA\boldsymbol{f}'\right) \\ &= y' \cdot \boldsymbol{f}' = \mathcal{D}_{x'}(y')f'(x').\end{aligned}$$

□

Proposition 3.2. *For $f(x) \in K[x]$, we have*

$$f(x+ty) = \sum_{j=0}^{\infty} t^j f^{(j)}(x,y),$$

for an indeterminate t.

Proof. This is a direct consequence of the Taylor expansion. □

Notation 3.3. *Let* $\boldsymbol{h} = (h_1(x), h_2(x), \ldots, h_n(x))$ *be a system of homogeneous polynomials* $h_i \in K[x_1, \ldots, x_n]$. *We define the differential operator* $\mathcal{D}_x(\boldsymbol{h}) : K(x) \to K(x)$ *associated to* $\boldsymbol{h}$ *by*

$$\mathcal{D}_x(\boldsymbol{h})f(x) = \sum_{j=1}^{n} h_j(x)\frac{\partial f(x)}{\partial x_j} = f^{(1)}(x, h(x)).$$

Furthermore we denote by

$$\mathrm{Sol}(\boldsymbol{h}; K[x])$$

the set of solutions in $K[x]$ *of the differential equation*

$$\mathcal{D}_x(\boldsymbol{h})f(x) = 0.$$

Namely,

$$\mathrm{Sol}(\boldsymbol{h}; R) = \left\{ f(x) \in K[x_1, \ldots, x_n] \;\middle|\; \sum_{j=1}^{n} h_j(x)\frac{\partial f(x)}{\partial x_j} = 0 \right\}.$$

Note that $\mathrm{Sol}(\boldsymbol{h}; K[x])$ *is a graded subalgebra of* $K[x]$.

Definition 3.4. *A system* $\boldsymbol{h} = (h_1(x), \ldots, h_n(x))$ *of polynomials is called self-vanishing, if* $h_j(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$ *for all* $j = 1, 2, \ldots, n$. *In addition to it, if* $\mathrm{GCD}(h_1, \ldots, h_n) = 1$, *we will say that* $\boldsymbol{h}$ *is a reduced self-vanishing system.*

Example 3.5. A constant vector $\boldsymbol{h} = (c_1, c_2, \ldots, c_n) \in K^n$ is obviously a self-vanishing system.

Example 3.6. Let $h_j \in K[x]$ be homogeneous polynomials (of the same degree). Suppose that $\boldsymbol{h} = (h_1, \ldots, h_n)$ satisfy the following conditions.

1. $h_1 = \cdots = h_r = 0,$ for some integer $r;\ 1 \le r < n.$
2. The polynomials $h_{r+1}, \ldots, h_n$ do not involve the variables $x_{r+1}, \ldots, x_n$.

Then $\boldsymbol{h}$ is a self-vanishing system of forms.

Definition 3.7. Let $\boldsymbol{f} = (f_1, \ldots, f_n)$ be a system of forms in $K[x]$. We denote by $\partial\boldsymbol{f}/\partial x_j$ the system of forms:

$$\frac{\partial \boldsymbol{f}}{\partial x_j} = \frac{\partial}{\partial x_j}\boldsymbol{f} = \left(\frac{\partial f_1}{\partial x_j}, \frac{\partial f_2}{\partial x_j}, \ldots, \frac{\partial f_n}{\partial x_j}\right).$$

This should not be confused with the notation already used:

$$\frac{\partial f(x)}{\partial x} := \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Proposition 3.8. *Let* $\boldsymbol{f} := (f_1, f_2, \ldots, f_n)$ *be a system of forms in* $K[x]$, *in which the components are algebraically dependent. Let* $y = (y_1, \ldots, y_n)$ *be a coordinate system algebraically independent of* x. *Let* $\phi : K[y] \to K[x]$ *be the homomorphism defined by*

$$y_j \mapsto f_j, (j = 1, 2, \ldots, n).$$

Let $g = g(y) \in \ker\phi$ *be a non-zero homogeneous polynomial of the least degree in* $\ker\phi$. *As in Proposition 2.3, define* $h_j \in K[x]$ *by*

$$h_j = \frac{\partial g}{\partial y_j}(f_1, \ldots, f_n).$$

Let $\boldsymbol{h} = (h_1, \ldots, h_n)$. *Then*

(a) $\boldsymbol{h}$ *is a syzygy of* $\boldsymbol{f}$.

(b) $\boldsymbol{h}$ *is a syzygy of* $\partial\boldsymbol{f}/\partial x_j$ *for every* $j = 1, 2, \ldots, n$.

Proof. Since $g = g(y)$ is homogeneous, we have

$$\frac{\partial g}{\partial y_1}y_1 + \frac{\partial g}{\partial y_2}y_2 + \cdots + \frac{\partial g}{\partial y_n}y_n = (\deg g)g.$$

Make the substitution $y_i \mapsto f_i$. Then we have

$$h_1f_1 + h_2f_2 + \cdots + h_nf_n = 0.$$

This shows the first assertion. By definition of $g = g(y)$, we have $g(f_1, f_2, \ldots, f_n) = 0$. Apply the operator $\frac{\partial}{\partial x_j}$ to the this equality. Then we have

$$0 = \frac{\partial g(f_1, f_2, \ldots, f_n)}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial g}{\partial y_k}(f_1, \ldots, f_n)\frac{\partial f_k}{\partial x_j} = \sum_{k=1}^{n} h_k \frac{\partial f_k}{\partial x_j}$$

This shows the second assertion. □

Theorem 3.9. *Let* $f = f(x) \in K[x]$ *be a homogeneous polynomial and put* $f_j = \frac{\partial f}{\partial x_j}$. *Assume that* $f_1, \ldots, f_n$ *are algebraically dependent. Let* $\boldsymbol{f} = (f_1, \ldots, f_n)$ *and let* $\boldsymbol{h} = (h_1, \ldots, h_n)$ *be a system of forms as defined in Proposition 3.8 for* $\boldsymbol{f}$. *Then*

(a) $f(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$.

(b) $f_j(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$ *for* $j = 1, 2, \ldots, n$.

(c) $f(x) \in \mathrm{Sol}(\partial\boldsymbol{h}/\partial x_j; K[x])$ *for* $j = 1, 2, \ldots, n$.

(d) $\boldsymbol{h}$ *is a self-vanishing system of forms.*

Proof. The assertion (a) follows immediately from Proposition 3.8 (a). Proposition 3.8 (b) says that $\boldsymbol{h}$ is a syzygy of $\partial\boldsymbol{f}/\partial x_j$. This means that

$$\sum_{k=1}^{n} h_k \frac{\partial f_k}{\partial x_j} = 0.$$

But $f_k = \frac{\partial f}{\partial x_k}$. Hence we have

$$\sum_{k=1}^{n} h_k \frac{\partial f_k}{\partial x_j} = \sum_{k=1}^{n} h_k \frac{\partial f_j}{\partial x_k} = 0.$$

This shows assertion (b).

Again by Proposition 3.8 (a) we have $\boldsymbol{h} \cdot \boldsymbol{f} = 0$. For each j, we have

$$\frac{\partial}{\partial x_j}(\boldsymbol{h} \cdot \boldsymbol{f}) = \frac{\partial \boldsymbol{h}}{\partial x_j} \cdot \boldsymbol{f} + \boldsymbol{h} \cdot \frac{\partial \boldsymbol{f}}{\partial x_j} = 0.$$

Again by Proposition 3.8 (b) we have $\boldsymbol{h} \cdot \partial\boldsymbol{f}/\partial x_j = 0$. Hence $\partial\boldsymbol{h}/\partial x_j \cdot \boldsymbol{f} = 0$. This shows that $f(x) \in \mathrm{Sol}(\frac{\partial}{\partial x_j}\boldsymbol{h}; K[x])$. Thus (c) is proved.

Since $\mathrm{Sol}(\boldsymbol{h}; K[x])$ is a commutative ring, we have

$$K[f_1, \ldots, f_n] \subset \mathrm{Sol}(\boldsymbol{h}; K[x]).$$

Since $h_j(x)$ are polynomials in $f_1, \ldots, f_n$, this shows that $h_j \in \mathrm{Sol}(\boldsymbol{h}, K[x])$. Thus (d) is proved.

□

Proposition 3.10. *Let $K[x] = K[x_1, \ldots, x_n]$ be the polynomial ring and let $\boldsymbol{h}$ be a self-vanishing system of forms in $K[x]$. Then, for any $f(x) \in K[x]$, we have*

$$\mathcal{D}_x(\boldsymbol{h}) \left(f^{(i)}(x, \boldsymbol{h}) \right) = (i+1) f^{(i+1)}(x, \boldsymbol{h}).$$

Proof. Let $y = (y_1, y_2, \ldots y_n)$ be a coordinate system independent of x. Write for a moment $F(x, y) = f^{(i)}(x, y)$. Note that

$$\frac{\partial F(x, \boldsymbol{h})}{\partial x_j} = \left\{ \frac{\partial F(x, y)}{\partial x_j} \right\}_{y \mapsto \boldsymbol{h}} + \left\{ \sum_{k=1}^{n} \frac{\partial F(x, y)}{\partial y_k} \frac{\partial h_k}{\partial x_j} \right\}_{y \mapsto \boldsymbol{h}}, \tag{2}$$

and

$$\mathcal{D}_x(\boldsymbol{h}) F(x, \boldsymbol{h}) = \sum_{j=1}^{n} h_j(x) \left\{ \frac{\partial F(x, y)}{\partial x_j} \right\} \Bigg|_{y \mapsto \boldsymbol{h}} + \sum_{j=1}^{n} h_j(x) \left\{ \sum_{k=1}^{n} \frac{\partial F(x, y)}{\partial y_k} \frac{\partial h_k(x)}{\partial x_j} \right\} \Bigg|_{y \mapsto \boldsymbol{h}}, \tag{3}$$

where $y \mapsto \boldsymbol{h}$ means substitution. Also note that $f^{(i+1)}(x, \boldsymbol{h}) = (1/(i+1)) \left\{ \mathcal{D}_x(y) F(x, y) \right\}_{y \mapsto \boldsymbol{h}}$. Since $\mathcal{D}_x(\boldsymbol{h}) h_k(x) = \sum_{j=1}^{n} h_j(x) \frac{\partial h_k(x)}{\partial x_j} = 0$ for every $k = 1, 2, \ldots, n$, the second summand of RHS in equation (3) reduces to 0. □

Theorem 3.11. *Suppose that $\boldsymbol{h} = (h_1, \ldots, h_n)$ is a self-vanishing system of forms in $K[x]$. Then, for a homogeneous polynomial $f(x) \in K[x]$, the following conditions are equivalent:*

(a) $f(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$.
(b) $f^{(j)}(x, \boldsymbol{h}) = 0$, *for* $j = 1, 2, \ldots$
(c) $f(x + t\boldsymbol{h}(x)) = f(x)$ *for any* $t \in K'$, *where* K' *is any extension field of* K.

Proof. (a) $\Leftrightarrow$ (b) By Proposition 3.10, it is enough to show the case for $j = 1$.

$$\begin{aligned} f^{(1)}(x, \boldsymbol{h}) &= \left\{ \mathcal{D}_x(y) f(x) \right\}_{y \mapsto \boldsymbol{h}} \\ &= \mathcal{D}_x(\boldsymbol{h}) f(x) \\ &= h_1 \frac{\partial f}{\partial x_1} + h_2 \frac{\partial f}{\partial x_2} + \cdots + h_n \frac{\partial f}{\partial x_n} = 0. \end{aligned}$$

The equivalence of (b) and (c) follows immediately from Proposition 3.2. □

Corollary 3.12. *Let $\boldsymbol{h}$ be a self-vanishing system in $K[x]$. If $f(x), g(x) \in K[x] \setminus \{0\}$, and if $f(x)g(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$, then both $f(x), g(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$.*

Proof. Using Taylor expansion (Proposition 3.2), we have

$$f(x + t\boldsymbol{h}) g(x + t\boldsymbol{h}) = \sum_{j=0}^{\infty} \sum_{k+l=j} t^{k+l} f^{(k)}(x, \boldsymbol{h}) g^{(l)}(x, \boldsymbol{h}).$$

Let k_0 be the lowest degree for which $f^{(k_0)} \neq 0$, and similarly l_0 for $g^{(l)}(x, \boldsymbol{h})$. Then $f(x)g(x) \in \mathrm{Sol}(\boldsymbol{h}, R)$ implies $k_0 + l_0 = 0$. Hence $k_0 = l_0 = 0$. Again by Theorem 3.11, proof is complete. □

Corollary 3.13. *Suppose that $\boldsymbol{h} = (h_1, \ldots, h_n)$ is a self-vanishing system in $R = K[x]$ such that $h_n(x) \neq 0$. Put*

$$s_i(x) = x_i - \frac{h_i(x)}{h_n(x)} x_n, \ (1 \leq i \leq n).$$

Let $f(x) \in \mathrm{Sol}(\boldsymbol{h}; R)$. *Then*

$$f(x) = f(s_1, s_2, \ldots, s_{n-1}, 0),$$

and

$$\mathrm{Sol}(\boldsymbol{h}, K[x]) = K[s_1, s_2, \ldots, s_{n-1}] \cap K[x].$$

Proof. It is easy to check by direct computation that

$$\mathcal{D}_x(\boldsymbol{h}) \left(\frac{1}{h_n(x)} \det \begin{pmatrix} x_i & x_n \\ h_i(x) & h_n(x) \end{pmatrix} \right) = 0.$$

Hence $\mathcal{D}_x(\boldsymbol{h}) s_j(x) = 0$. This shows that

$$K[s_1, s_2, \ldots, s_{n-1}] \cap K[x] \subset \mathrm{Sol}(\boldsymbol{h}; K[x]).$$

To show the converse, let $f(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$. Then $f(x) = f(x + t\boldsymbol{h})$ for an indeterminate t by Theorem 3.11. Replace t for $t = -\frac{x_n}{h_n(x)}$. Then we get

$$f(x) = f(s_1, s_2, \ldots, s_{n-1}, 0).$$

□

Corollary 3.14. *Let* $\boldsymbol{h} = (h_1, h_2, \ldots, h_n)$ *be a self-vanishing system of polynomials. Let* $f(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$. *Suppose that* $f(x)$ *is homogeneous of positive degree* d. *Then we have*

$$f(h_1, h_2, \ldots, h_n) = 0.$$

In particular, if h_j *has positive degree,*

$$h_j(h_1, h_2, \ldots, h_n) = 0 \text{ for all } j = 1, 2, \ldots, n.$$

Proof. Let $y = (y_1, \ldots, y_n)$ be a coordinate system independent of x, and put $\mathcal{D}_x(y)^j f(x) = \frac{1}{j!} f^{(j)}(x, y)$. We have shown that $f^{(d)}(x, y) = f(y)$. In this equation substitute y for $\boldsymbol{h}$. Then we have $f^{(d)}(x, \boldsymbol{h}) = f(\boldsymbol{h})$. In Theorem 3.11, we showed that $f^{(j)}(x, \boldsymbol{h}) = 0$ for $j > 0$. Thus we have $f(\boldsymbol{h}) = 0$. □

4 Forms with zero Hessian and reduced self-vanishing systems

Proposition 4.1. *Let* $R = K[x_1, \ldots, x_n]$. *Let* $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)$ *be a system of forms in* R. *Then* $\mathrm{rank}_{K(x)} \left(\frac{\partial f_i}{\partial x_j} \right) = \mathrm{tr.deg}_K\, K(f_1, f_2, \ldots, f_n)$. *In particular the following conditions are equivalent.*

1. $f_1, \ldots, f_r$ *are algebraically dependent.*
2. *The rank of Jacobian matrix* $(\frac{\partial f_i}{\partial x_j})$ *is* $< r$.
3. $\mathrm{tr.deg}_K\, K(f_1, f_2, \ldots, f_r) < r$.

Proof. Left to the reader. □

We say that a homogeneous polynomial $f \in R = K[x_1, \ldots, x_n]$ has zero Hessian if $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 0$. (Whenever we discuss forms f with zero Hessian, we assume $\deg f \geq 2$.) Let f be a homogeneous form in $K[x_1, \ldots, x_n]$. Since the Hessian determinant of f is the Jacobian determinant of the partial derivatives of f, the following proposition follows immediately from Lemma 4.1.

Proposition 4.2. *A homogeneous polynomial $f \in R$ is a form with zero Hessian if and only if the partial derivatives*

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$$

are algebraically dependent.

Definition 4.3. *Let $y = (y_1, \ldots, y_n)$ be a coordinate system independent of x. Let $f \in K[x]$ be a form with zero Hessian, and let*

$$\phi : K[y_1, \ldots, y_n] \to K[x_1, \ldots, x_n]$$

be the homomorphism defined by $\phi(y_j) = \frac{\partial f}{\partial x_j}$. We denote by $\mathcal{I}(f)$ the kernel of ϕ. (By Proposition 4.2, $\mathcal{I}(f) \neq 0$.)

Definition 4.4. Let $f \in R = K[x]$ be a form with zero Hessian. Let $\mathcal{I}(f) \subset K[y]$ be as in Definition 4.3. Put $f_j = \frac{\partial f}{\partial x_j}$. Let $g(y) = g(y_1, \ldots, y_n) \in \mathcal{I}(f)$ be a homogeneous form of the least degree in $\mathcal{I}(f)$. Let

$$h_i'(x_1, x_2, \ldots, x_n) = \frac{\partial g}{\partial y_i}(f_1, \ldots, f_n),$$

$$h_i(x_1, x_2, \ldots, x_n) = \frac{1}{\mathrm{GCD}(h_1', h_2', \ldots, h_n')} h'(x_1, \ldots, x_n).$$

We call the vector $\boldsymbol{h}' := (h_1', h_2', \ldots, h_n')$ a system of polynomials arising from $f(x)$, and $\boldsymbol{h} := (h_1, \ldots, h_n)$ a reduced system of polynomials arising from $f(x)$.

Remark 4.5. By Remark 2.4, $\boldsymbol{h} \neq 0$ as well as $\boldsymbol{h}' \neq 0$.

Remark 4.6. Assume that $\mathrm{rank}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = n - 1$. Then the ideal $\mathcal{I}(f)$ is a principal ideal of $K[y]$. In this case $g \in \mathcal{I}(f) \setminus \{0\}$ with the smallest degree is uniquely determined (up to a constant multiple). Hence $\boldsymbol{h}$, in Definition 4.4, is uniquely determined by f. On the other hand by Proposition 3.8 (b), we see that $\boldsymbol{h}$ is a null vector of the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$. Such a polynomial vector is unique up to a multiple of a polynomial. Hence, a system of forms $\boldsymbol{h}$ is the self-vanishing system as defined in Definition 4.4 if and only if the following two conditions are satisfied.

1. $(h_1, h_2, \ldots, h_n)\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 0$,
2. $\mathrm{GCD}(h_1, h_2 \ldots, h_n) = 1$.

Theorem 4.7 (Gordan-Noether)**.** *Suppose that $f(x) \in K[x_1, \ldots, x_n]$ is a form with zero Hessian. Then a variable can be eliminated by means of a birational transformation of the variables.*

Proof. Let $\boldsymbol{h}$ be a reduced self-vanishing system arising from f. Then we have $f(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$ by Theorem 3.9. By Lemma 2.1 and Remark 2.4, we may assume that $h_n \neq 0$. Put $s_j = x_j - \frac{h_j}{h_n} x_n$, for $j = 1, \ldots, n-1$. Set $s_n = 0$. Then by Corollary 3.13, f is a polynomial in $s_1, \ldots, s_{n-1}$. We

claim that

$$K(s_1, \ldots, s_{n-1}, x_n) = K(x_1, \ldots, x_n).$$

In fact we have

$$x_j = s_j + \frac{h_j(x)}{h_n(x)} x_n, \quad j = 1, 2, \ldots, n-1.$$

Since $h_j(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$, we have $h_j(x + t\boldsymbol{h}(x)) = h_j(x)$ for any t in any extension field of K by Theorem 3.11 (c). Now let $t = -\frac{x_n}{h_n}$. Then $h_j(s) = h_j(x)$. This show that $x_j \in K(s_1, \ldots, s_{n-1}, x_n)$ for all j, as desired. □

Proposition 4.8. *Suppose that $f(x) \in K[x]$ is a form with zero Hessian. Let $\mathcal{I}(f(x))$ be the ideal of $K[y]$ as defined in Definition 4.3. Then the following conditions are equivalent.*

(a) *The ideal $\mathcal{I}(f(x))$ contains a linear form.*
(b) *The partial derivatives of $f(x)$ are linearly dependent.*
(c) *A variable can be eliminated from $f(x)$ by means of a linear transformation of the variables.*

Proof. The equivalence of (a) and (b) is clear. Suppose that there exists a non-trivial relation

$$a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = 0,$$

where $f_j = \frac{\partial f}{\partial x_j}$ and $a_j \in K$. It is possible to choose a set of linearly independent linear forms $y_1, \ldots, y_n$ in $x_1, \ldots, x_n$ such that $\frac{\partial x_j}{\partial y_1} = a_j$. Then

$$\frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_1} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_1} = 0.$$

This shows that if f is expressed in terms of y_j, then f does not contain y_1. Thus (b) $\Rightarrow$ (c). The same argument shows (b) $\Leftarrow$ (c) as well. □

Theorem 4.9. *Let $f(x) \in K[x]$ be a form with zero Hessian and let $\boldsymbol{h}' = (h'_1, \ldots, h'_n)$ a self-vanishing system of polynomials associated to an element $g = g(y_1, \ldots, y_n) \in \mathcal{I}(f(x))$ of the least degree. Similarly let $\boldsymbol{h} = (h_1, \ldots, h_n)$ be the reduced system of polynomials defined by*

$$\boldsymbol{h} = \frac{1}{\mathrm{GCD}(\boldsymbol{h}')}\boldsymbol{h}'.$$

(See Definition 4.4.) Then we have:

(a) *$\boldsymbol{h}$ and $\boldsymbol{h}'$ are self-vanishing systems of polynomials.*
(b) *$f(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$.*
(c) *$\frac{\partial}{\partial x_j} f(x) \in \mathrm{Sol}(\boldsymbol{h}; K[x])$, for all $j = 1, 2, \ldots, n$.*

Proof. (a) By Theorem 3.9, $\boldsymbol{h}'(x)$ is a self-vanishing system. Note that $\mathrm{Sol}(\boldsymbol{h}'; K[x]) = \mathrm{Sol}(\boldsymbol{h}; K[x])$. By Corollary 3.12, $\boldsymbol{h}(x)$ is a self-vanishing system. (b) and (c) are proved in Theorem 3.9. □

5 Binary and ternary forms with zero Hessian

Theorem 5.1. *Assume that $n = 2$ and let $f \in K[x_1, x_2]$ be a form of degree d with zero Hessian. Then $f = (a_1x_1 + a_2x_2)^d$ for some $a_1, a_2 \in K$.*

Proof. Let $\mathcal{I}(f) \subset K[y_1, y_2]$ be the ideal defined in Definition 4.3. Since $\mathcal{I}(f)$ is a prime ideal, it is a principal ideal generated by a linear form. Hence the assertion follows from Proposition 4.8. □

Lemma 5.2. *Let $\boldsymbol{h} = (h_1, \ldots, h_n)$ be a system of forms in $K[x_1, \ldots, x_n]$. Then, if $\boldsymbol{h}$ is self-vanishing, we have* rank $\left(\frac{\partial h_i}{\partial x_j}\right) \leq n/2$. *In particular* Krull.dim $K[h_1, \ldots, h_n] \leq n/2$.

Proof. Consider the morphism of the affine space

$$\phi : \mathbb{A}^n \xrightarrow{\phi} \mathbb{A}^n,$$

defined by $x_j \to h_j(x)$. Since $\boldsymbol{h}$ is self-vanishing, the composition is zero: $\phi^2 = 0$. Let $J_\phi = (\frac{\partial h_i}{\partial x_j})$ be the Jacobian matrix. Since we have $J_\phi^2 = 0$, the rank cannot exceed $n/2$. □

Theorem 5.3. *Suppose that $f = f(x) \in K[x_1, x_2, x_3]$ is a form with zero Hessian. Then a variable can be eliminated from f by means of a linear transformation of variables.*

Proof. Let $\boldsymbol{h} = (h_1, h_2, h_3)$ be a reduced system of polynomials arising from $f(x)$ (See Definition 4.4). We claim that $\boldsymbol{h}$ is a constant vector. Suppose that it is not. Then Krull.dim $K[h_1, h_2, h_3]$ $= 1$ by Proposition 5.2. Thus any two of the elements h_1, h_2, h_3 has a homogeneous algebraic relation. Since $K[h_1, h_2, h_3]$ is an integral domain, and K is algebraically closed, they should be linear relations. Thus we have $\dim_K Kh_1 + Kh_2 + Kh_3 = 1$. Since $\mathrm{GCD}(h_1, h_2, h_3) = 1$, this is impossible unless they are constants. Recall that we have

$$h_1\frac{\partial f}{\partial x_1} + h_2\frac{\partial f}{\partial x_2} + h_3\frac{\partial f}{\partial x_3} = 0.$$

By Proposition 4.8, proof is complete. □

6 The rational map defined by $\boldsymbol{h}$

For the rest of this section we assume that $n \geq 4$. Let $R = K[x_1, \ldots, x_n]$ and let $f = f(x_1, \ldots, x_n)$ be a form with zero Hessian. Let $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)$ with $f_j = \frac{\partial f}{\partial x_j}$ and let $\boldsymbol{h} = (h_1, \ldots, h_n)$ be a system of forms associated to $\boldsymbol{f}$ as defined in Definition 4.4.

Let

$$Z : \mathbb{P}^{n-1}(x) \to \mathbb{P}^{n-1}(y)$$

be the rational map defined by the correspondence $x = (x_1 : \cdots : x_n) \mapsto (h_1 : \cdots : h_n)$. Let W be the image of Z and T the fundamental locus of Z in $\mathbb{P}^{n-1}(x)$ defined by the equations $h_1(x) = h_2(x) = \cdots = h_n(x) = 0$. The algebraic set $W \subset \mathbb{P}^{n-1}(y)$ is defined by $\mathcal{I}$ as in Proposition 3.8.

Proposition 6.1. *The following conditions are equivalent.*

(a) $\deg(h_j) = 0$, *i.e.,* $\boldsymbol{h}$ *is a constant vector.*

(b) $\dim W = 0$, *i.e.,* W *is a one-point set.*

(c) T *is empty, i.e.,* Z *is a morphism.*

Proof. Suppose that $\boldsymbol{h}$ is not a constant. Then we have

$$h_j(h_1, \ldots, h_n) = 0 \text{ for every } j = 1, \ldots, n.$$

by Corollary 3.14. Thus any specialization of $(h_1, \ldots, h_n)$ is a point of T. This shows that if T is empty, then $\boldsymbol{h}$ is a constant vector. All other implications are trivial. □

Proposition 6.2. *If* $\dim W \geq 1$*, then*

$$n/2 \leq \text{Krull.dim } K[x]/(h_1, \ldots, h_n) \leq n - 2,$$

or equivalently,

$$n/2 - 1 \leq \dim T \leq n - 3.$$

Proof. Since T is not empty, the ideal $(h_1, h_2, \ldots, h_n) \subset K[x]$ is not the unit ideal by the previous proposition. On the other hand, since $\text{GCD}(h_1, \ldots, h_n) = 1$, it is not a principal ideal. Hence $\text{ht}(h_1, \ldots, h_n) \geq 2$. This shows that $\dim T = \text{Krull.dim } K[x]/(h_1, \ldots, h_n) - 1 \leq n - 3$. On the other hand there exists a surjective homomorphism of rings:

$$K[x_1, \ldots, x_n]/(h_1, \ldots, h_n) \to K[h_1, \ldots, h_n],$$

$$x_j \mapsto h_j,$$

provided that $\deg \boldsymbol{h} > 0$. Note that $K[x_1, \ldots, x_n]/(h_1, \ldots, h_n)$ is the fiber at the origin of the inclusion map $K[h_1, \ldots, h_n] \to K[x_1, \ldots, x_n]$ and $\text{Krull.dim } K[h_1, \ldots, h_n] \leq n/2$ by Lemma 5.2. Hence we have $\text{Krull.dim } K[x_1, \ldots, x_n]/(h_1, \ldots, h_n) \geq n/2$. □

For the rest of this section we assume that $\dim W = 1$. In this case the fiber of $Z : \mathbb{P}^{n-1}(x) \to \mathbb{P}^{n-1}(y)$ is a hypersurface of $\mathbb{P}^{n-1}(x)$. Thus, for any $\omega \in \mathbb{P}^{n-1}(y)$, $Z^{-1}(\omega)$ is defined by one homogeneous polynomial in $x = (x_1, \ldots, x_n)$.

Definition 6.3. Let $Z : \mathbb{P}^{n-1}(x) \to \mathbb{P}^{n-1}(y)$ be as above. Let $\omega = (\omega_1 : \omega_2 : \cdots : \omega_n) \in \mathbb{P}^{n-1}(y)$. We denote by $g^{(\omega)}(x)$ the homogeneous polynomial in $K[x]$ that defines the hypersurface of the fiber of Z at $\omega \in W \subset \mathbb{P}^{n-1}(y)$.

For each point $\omega = (\omega_1 : \omega_2 : \cdots : \omega_n) \in \mathbb{P}^{n-1}$, we define the differential operator $\mathcal{D}_x(\omega)$ on $R = K[x_1, \ldots, x_n]$ by

$$\mathcal{D}_x(\omega) f(x) = \sum_{j=1}^{n} \omega_j \frac{\partial f(x)}{\partial x_j}$$

whose value is determined up to a non-zero constant factor. Hence we may speak of the set of solutions of the equation $\mathcal{D}_x(\omega) f(x) = 0$. We denote the space of solutions of $\mathcal{D}_x(\omega) f(x) = 0$ in $R = K[x_1, \ldots, x_n]$ by $\text{Sol}(\omega; R)$. It is a subring of R. For any subset U of $\mathbb{P}^{n-1}$ we denote by $\text{Sol}(U; R)$ the space of solutions of the system of linear differential equations

$$\mathcal{D}_x(\omega)(f(x)) = 0,\ \omega \in U.$$

If we denote by $L(U)$ the linear closure of U in $\mathbb{P}^{n-1}$ it is easy to see that

$$\mathrm{Sol}(U; R) = \mathrm{Sol}(L(U); R).$$

If $U = \{\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(s)}\}$ is a finite set, then $\mathrm{Sol}(U; R)$ is also denoted as

$$\mathrm{Sol}(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(s)}; R).$$

The same notation is used if we replace $\mathbb{P}^{n-1}(y)$ for the vector space K^n in the obvious sense. Namely for a linear subspace L of K^n, we denote by $\mathrm{Sol}(L; R)$, the set of solutions of the differential equations

$$\mathcal{D}_x(\boldsymbol{a})F(x) = 0 \text{ for all } \boldsymbol{a} \in L,$$

where $\boldsymbol{a}$ denotes a row vector in L regarded as a system of constants. In a set theoretic notation,

$$\mathrm{Sol}(L; K[x]) = \left\{ F(x) \in R \,\middle|\, \left(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + \cdots + a_n\frac{\partial}{\partial x_n} \right) F(x) = 0, (a_1, \ldots, a_n) \in L \right\}.$$

Note that $\mathrm{Sol}(L; R)$ is a subring of R generated by homogeneous linear forms.

The relation between the subring $\mathrm{Sol}(L; R)$ of R and the ideal I for the linear subspace in $\mathbb{P}^{n-1}$ is very important for us. In the next theorem and corollary we describe a set of generators of the subring $\mathrm{Sol}(L; R)$ and a set of generators that defines the linear space L as a subspace of $\mathbb{P}^{n-1}$.

Theorem 6.4. *Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over K. Let $A = (a_i^{(j)})$ be a $k \times n$ matrix, $a_i^{(j)} \in K$, with $\boldsymbol{a}^{(j)} = (a_1^{(j)}, a_2^{(j)}, \ldots, a_n^{(j)})$ as the j-th row. Suppose that the rows are linearly independent. Assume that $k < n$. Then the set of solutions as a subring of $K[x]$*

$$\mathrm{Sol}(\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(k)}; R) := \bigcap_{j=1}^{k} \mathrm{Sol}(\boldsymbol{a}^{(j)}; R)$$

is isomorphic to the polynomial ring in $n-k$ variables. It is generated by the $(k+1) \times (k+1)$ minors of the matrix

$$A' := \left(\begin{array}{ccccc} a_1^{(1)} & a_2^{(1)} & \cdots & a_{n-1}^{(1)} & a_n^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \cdots & a_{n-1}^{(2)} & a_n^{(2)} \\ & & & & \\ a_1^{(k)} & a_2^{(k)} & \cdots & a_{n-1}^{(k)} & a_n^{(k)} \\ \hline x_1 & x_2 & \cdots & x_{n-1} & x_n \end{array} \right)$$

Proof. Note that if $k = 1$, then the assertion is proved by Corollary 3.13, since a constant vector is a self-vanishing system. Also note that row operations of the matrix A' does not change the set of solutions. Thus one sees easily that the subring we are looking for is generated by $n-k$ linearly independent homogeneous linear forms.

The assertion in the general case follows from the fact that the $(k+1) \times (k+1)$ minors $\Delta_{j_1 j_2 \cdots j_k}$ of A' are solutions of the differential equations and those linear forms span a subspace of linear forms of rank $n-k$. (To be strict one uses Plücker relations to show they span a subspace of dimension $n-k$. Details are left to the reader. Cf.[1] [11].) □

Corollaries 6.5 and 6.6 that follow are other ways to describe the set of linear forms as generators for $\mathrm{Sol}(L;R)$.

Corollary 6.5. *Let R and A be the same as Theorem 6.4, and furthermore let L be the vector subspace in K^n generated by the rows of A. Then the set of solutions $\mathrm{Sol}(L;R)$, as a subring of $R=K[x_1,\dots,x_n]$, is generated by the linear forms $l=l(x)$ such that*

$$l(a_1^{(j)},a_2^{(j)},\dots,a_n^{(j)})=0,$$

for all $j=1,2,\dots,k$.

Proof. Left to the reader. □

Corollary 6.6. *Let*

$$i:\mathbb{P}^{n-1}(y)\to\mathbb{P}^{n-1}(x)$$

be the natural isomorphism that sends $y_j\mapsto x_j$, for $j=1,2,\dots,n$. Suppose that $U\subset\mathbb{P}^{n-1}(y)$ is a subset and let $L(U)$ be the linear closure of U. Then $\mathrm{Sol}(U;R)=\mathrm{Sol}(L(U);R)$ is generated by the linear forms

$$\{\ l(x)\in K[x_1,x_2,\dots,x_n]|l(\omega_1,\omega_2,\dots,\omega_n)=0\ \ \textit{for all}\ (\omega_1:\omega_2:\cdots:\omega_n)\in L(U)\},$$

as a subring of $K[x]$.

The following two theorems are very important to determine the forms in four and five variables with zero Hessian.

Theorem 6.7. *Let $\boldsymbol{h}=(h_1,\dots,h_n)$ be a self-vanishing system of forms in $R=K[x_1,\dots,x_n]$. Let $Z:\mathbb{P}^{n-1}(x)\to\mathbb{P}^{n-1}(y)$ be the rational map defined by $y_j=h_j(x)$. Let $T\subset\mathbb{P}^{n-1}(x)$ be the fundamental locus and W the image of Z. Assume that* $\dim W=1$. *Let*

$$i:\mathbb{P}^{n-1}(y)\to\mathbb{P}^{n-1}(x)$$

be the natural map $y_j\to x_j$. Then $i(L(W))\subset T$, where $L(W)$ is the linear closure of W in $\mathbb{P}^{n-1}(y)$.

Proof. Recall that T is defined by the polynomials $h_1(x),\dots,h_n(x)$. On the other hand $i(L(W))$ is defined by linear forms which vanish on the set $i(L(W))$. Hence, in view of Corollary 6.6, this follows immediately from Theorem 6.8 below. □

Theorem 6.8. *With the same notation and assumption in Theorem 6.7, we have*

$$h_j(x)\in\mathrm{Sol}(i(L(W));R)\ \textit{for all}\ j=1,2,\dots,n.$$

We prove it after some propositions. For the rest of this section we fix notation and assumption of Theorem 6.7. In particular it is assumed that $\dim W=1$.

Proposition 6.9. *For any $\omega\in W$, we have $g^{(\omega)}(x)\in\mathrm{Sol}(\boldsymbol{h};R)$.*

Proof. Choose a hyperplane $H\subset\mathbb{P}^{n-1}(y)$ such that $\omega\in H$, and suppose that H is defined by the linear equation

$$a_1y_1+\cdots+a_ny_n=0.$$

Put $f(x) = \sum_{j=1}^{n} a_j h_j(x)$. Since K is infinite, it is possible to choose H so that $f(x) \neq 0$. Then for any $\alpha \in Z^{-1}(\omega)$, if $\alpha \in T$, then $h_j(\alpha) = 0$ for all j by definition of T. Hence $f(\alpha) = 0$. If $\alpha \notin T$, then

$$f(\alpha) = \sum_{j=1}^{n} a_j h_j(\alpha) = c \sum_{j=1}^{n} a_j \omega_j = 0.$$

This shows that any point on $Z^{-1}(\omega)$ is a zero of $f(x)$. Hence $f(x)$ is a multiple of $g^{(\omega)}(x)$. By Corollary 3.12, this proves that $g^{(\omega)}(x) \in \mathrm{Sol}(\boldsymbol{h}; R)$ for any $\omega \in W$. □

Proposition 6.10. *For each point $\omega \in W$, we have $g^{(\omega)}(x) \in \mathrm{Sol}(L(W); R)$.*

Proof is preceded by two lemmas.

Lemma 6.11. *For $\omega, \omega' \in W$, the polynomial $\mathcal{D}_x(\omega') g^{(\omega)}(x)$ is divisible by $g^{(\omega')}(x)$.*

Proof. In Proposition 6.9, we proved that $g^{(\omega)}(x) \in \mathrm{Sol}(\boldsymbol{h}; R)$. So if $\mathcal{D}_x(\boldsymbol{h})$ is applied to $g^{(\omega)}(x)$, it becomes 0. Namely,

$$h_1(x) \frac{\partial g^{(\omega)}(x)}{\partial x_1} + \cdots + h_n(x) \frac{\partial g^{(\omega)}(x)}{\partial x_n} = 0.$$

Choose $\alpha \in Z^{-1}(\omega')$ and make the substitution $x \mapsto \alpha$ in the above equality. Then the left hand side is the same as

$$\mathcal{D}_x(\omega') g^{(\omega)}(x)$$

evaluated at $x = \alpha$. We have shown that the zero locus of $g^{(\omega')}(x)$ is contained in that of $\mathcal{D}_x(\omega') g^{(\omega)}(x)$. Since $(g^{(\omega')}(x))$ is reduced, this proves the assertion. □

Lemma 6.12. *For any point $\omega \in W$, we have $\mathcal{D}(\omega) g^{(\omega)}(x) = 0$.*

Proof. In the previous lemma let $\omega' = \omega$. Then we have proved that $\mathcal{D}(\omega) g^{(\omega)}(x) = r(x) g^{(\omega)}(x)$ for some $r(x) \in R$. For the degree reason, we get the assertion. □

Proof of Proposition 6.10. By Lemma 6.12, we have

$$\mathcal{D}(\omega) \left(\mathcal{D}(\omega') g^{(\omega)}(x) \right) = \mathcal{D}(\omega') \left(\mathcal{D}(\omega) g^{(\omega)}(x) \right) = 0.$$

By Lemma 6.11, we may write $\mathcal{D}(\omega') g^{(\omega)}(x) = r(x) g^{(\omega')}(x)$ for some $r(x)$. Since

$$\mathcal{D}(\omega) \left(r(x) g^{(\omega')}(x) \right) = 0,$$

$r(x) g^{(\omega')}(x) \in \mathrm{Sol}(\omega; R)$. Therefore, since a constant vector is a self-vanishing system, we have $g^{(\omega')}(x) \in \mathrm{Sol}(\omega; R)$ by Corollary 3.12. □

Proof of Theorem 6.8. We have to show that $h_j(x) \in \mathrm{Sol}(L(W), R)$. We may assume that $h_j \neq 0$. Let W_j be the intersection of W and the hyperplane defined by $y_j = 0$ in the space $\mathbb{P}^{n-1}(y)$, and let $H_j \subset \mathbb{P}^{n-1}(x)$ be the hypersurface defined by $h_j(x) = 0$. Then W_j is a finite set, since $\dim W = 1$ and $W \not\subset \{y_j = 0\}$. Thus, if we write $W_j = \{\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(s)}\}$, then

$$H_j \supset Z^{-1}(\omega^{(1)}) \cup Z^{-1}(\omega^{(2)}) \cup \cdots \cup Z^{-1}(\omega^{(s)}).$$

Conversely if $\alpha \in H_j \setminus T$, then $Z(\alpha) \in W_j$. Thus

$$H_j \setminus T \subset Z^{-1}(\omega^{(1)}) \cup Z^{-1}(\omega^{(2)}) \cup \cdots \cup Z^{-1}(\omega^{(s)}).$$

Since T has codimension at least 2, it in fact shows that

$$H_j \subset Z^{-1}(\omega^{(1)}) \cup Z^{-1}(\omega^{(2)}) \cup \cdots \cup Z^{-1}(\omega^{(s)}).$$

Thus we have proved that

$$h_j(x) = (\text{constant}) \times g^{(\omega^{(1)})}(x) \times g^{(\omega^{(2)})}(x) \times \cdots \times g^{(\omega^{(s)})}(x).$$

By Proposition 6.10, this completes the proof of Theorem 6.8. □

7 Quaternary and quinary forms with zero Hessian

Theorem 7.1. *Let $f = f(x_1, x_2, x_3, x_4) \in R \subset K[x_1, x_2, x_3, x_4]$ be a form with zero Hessian. Then f can be transformed into a form with three variable via a linear transformation of the variables x_1, x_2, x_3, x_4.*

Proof. Let $\boldsymbol{h} = (h_1, h_2, h_3, h_4)$ be a reduced self-vanishing system of forms arising from f. Then by Lemma 5.2, Krull.dim$K[h_1, h_2, h_3, h_4] \leq n/2 = 2$. This shows dim $W \leq 1$. Put $s = \dim L(W)$ (as a linear variety in $\mathbb{P}^3$). Note that $s+1 = \dim_K(Kh_1 + \cdots + Kh_4)$. Since $i(W) \subset T$ by Theorem 6.7 and dim $T \leq 1$ by Proposition 6.2, we have $s \leq 1$. If $s = 0$, we get the result by Proposition 6.1. If $s = 1$, we may assume that $h_1(x) = h_2(x) = 0$ by Lemmas 2.1 and 2.2. Let $g(y) \in K[y_1, \ldots, y_4]$ be the polynomial through which $h_1, \ldots, h_4$ are defined (cf. Definition 4.4). Then $h_1 = h_2 = 0$ implies that $\frac{\partial g}{\partial y_1} = \frac{\partial g}{\partial y_2} = 0$ by the minimality of the degree. (cf. Proposition 2.3 and Remark 2.4.) Hence g is a form only in two variables. Since g should be irreducible, g is a linear form. This implies that there exists a linear relation among the partial derivatives of f. Hence the assertion follows from Proposition 4.8. □

Proposition 7.2. *Let $f = f(x_1, x_2, x_3, x_4, x_5) \in R = K[x_1, x_2, x_3, x_4, x_5]$ be a form with zero Hessian. Let $\boldsymbol{h} = (h_1, h_2, h_3, h_4, h_5)$ be a self-vanishing system of polynomials associated to f as defined in Definition 4.4. Assume that there exists no linear relations among the partial derivatives of f. Then by a linear change of variables, f can be transformed into a form so that $h_1 = h_2 = 0$, and h_3, h_4, h_5 are polynomials only in x_1, x_2.*

Proof. Let $Z : \mathbb{P}^4(x) \to \mathbb{P}^4(y)$ be the rational map defined by $y_j = h_j(x)$ and let T be the fundamental locus and W the image of Z. Then, as the quaternary case, we have dim $W \leq 1$ by Lemma 5.2. If dim $W = 0$, a variable can be eliminated from f, since $\boldsymbol{h}$ is a constant vector by Proposition 6.1. Assume that dim $W = 1$. Then we have $i(L(W)) \subset T$ by Theorem 6.7. On the other hand we have dim $T \leq 2$ by Proposition 6.2. This shows dim $L(W) \leq 2$ as a linear subspace in $\mathbb{P}^4$. Let $s = \dim L(W)$ or equivalently $s + 1 = \dim_K \sum_{j=1}^5 Kh_j$. Since dim $W = 1$, $s \geq 1$. If $s = 1$, we may assume $h_1 = h_2 = h_3 = 0$. As in the proof of Theorem 7.1 this would imply that there exist a linear relation among the partial derivatives of f. Since we have excluded this case, we are left with the case $s = 2$. Then we may assume that $h_1 = h_2 = 0$ and $h_j \in \text{Sol}(i(L(W)); R)$ for $j = 3, 4, 5$. The linear subspace $i(L(W))$ consists of vectors $(0, 0, *, *, *)$. Hence h_3, h_4, h_5

should be polynomials only in x_1 and x_2.

□

Theorem 7.3. *Let $R = K[x_1, x_2, x_3, x_4, x_5]$. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ be a matrix with homogeneous entries $a_{ij} = a_{ij}(x_1, x_2) \in K[x_1, x_2]$ such that the 2×2 minors of A do not have a common factor. Put*

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ x_3 & x_4 & x_5 \end{vmatrix}.$$

Then any element in the algebra

$$S = K[x_1, x_2][\Delta]$$

is a polynomial with zero Hessian. Conversely, let f be a homogeneous form in five variables with zero Hessian and assume that f properly involves five variables. Then f can be transformed into a homogeneous polynomial in the ring S by means of a linear change of variables.

Proof. Recall $f(x) \in K[x]$ has zero Hessian if and only if the partial derivatives of f are algebraically dependent. (See Proposition 4.2.) Put $\boldsymbol{f} = (f_1, \ldots, f_5)$, where $f_j = \frac{\partial f}{\partial x_j}$. Let $\boldsymbol{h} = (h_1, \ldots, h_5)$ be a reduced self-vanishing system associated to $f(x)$. We have proved that $\boldsymbol{h} \cdot \frac{\partial}{\partial x_j}\boldsymbol{f} = 0, j = 1, \ldots, 5$ in Corollary 3.9. This shows that $f(x) \in \mathrm{Sol}(\frac{\partial}{\partial x_1}\boldsymbol{h}, \ldots, \frac{\partial}{\partial x_5}\boldsymbol{h}; R)$. By Proposition 7.2, we may assume that $h_1 = h_2 = 0$ and h_3, h_4, h_5 involve only x_1, x_2. Let $a_{ij} = a_{ij}(x_1, x_2)$ be the elements in $K[x_1, x_2]$ so that we have:

$$\begin{cases} \frac{\partial}{\partial x_1}\boldsymbol{h} = (0, 0, a_{11}, a_{12}, a_{13}) \\ \frac{\partial}{\partial x_2}\boldsymbol{h} = (0, 0, a_{21}, a_{22}, a_{23}) \end{cases}$$

Then $f(x) \in \mathrm{Sol}(\frac{\partial}{\partial x_1}\boldsymbol{h}, \frac{\partial}{\partial x_2}\boldsymbol{h}; R)$. Now $f(x) \in K[x_1, x_2, \Delta]$ follows from Theorem 7.4 below.

Conversely assume that $f \in S$. Notice that $f \in S$ satisfies

$$\begin{cases} a_{11}\frac{\partial f}{\partial x_3} + a_{12}\frac{\partial f}{\partial x_4} + a_{13}\frac{\partial f}{\partial x_5} = 0 \\ a_{21}\frac{\partial f}{\partial x_3} + a_{22}\frac{\partial f}{\partial x_4} + a_{23}\frac{\partial f}{\partial x_5} = 0. \end{cases}$$

Hence the polynomial vector

$$\left(\frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}, \frac{\partial f}{\partial x_5}\right)$$

is a multiple of

$$\left(\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right).$$

Thus the three partial derivatives $\partial f/\partial x_j$, $j = 3, 4, 5$ satisfy the same algebraic relation as the three minors of A. Hence f has zero Hessian. □

Theorem 7.4. *Let $R = K[u_1, u_2, x_1, x_2, x_3]$. Let S be the set of solutions of the following system of linear differential equations:*

$$\begin{cases} a_1^{(1)}\frac{\partial f}{\partial x_1} + a_2^{(1)}\frac{\partial f}{\partial x_2} + a_3^{(1)}\frac{\partial f}{\partial x_3} = 0 \\ a_1^{(2)}\frac{\partial f}{\partial x_1} + a_2^{(2)}\frac{\partial f}{\partial x_2} + a_3^{(2)}\frac{\partial f}{\partial x_3} = 0, \end{cases} \tag{4}$$

where the coefficients $a_j^{(i)} = a_j^{(i)}(u_1, u_2)$ *are elements of* $K[u_1, u_2]$. *Put* $\Delta = \begin{vmatrix} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} \\ x_1 & x_2 & x_3 \end{vmatrix}$.

Assume that the 2×2 *minors of the matrix*

$$\begin{pmatrix} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} \end{pmatrix}$$

generates a perfect ideal of height two in $K[u_1, u_2]$. *Then*

$$S = K[u_1, u_2, \Delta].$$

Proof. It is easy to see that $S \supset K[u_1, u_2, \Delta]$. Put $S' = K[u_1, u_2, \Delta]$. We have to prove that if $f = f(u_1, u_2, x_1, x_2, x_3) \in R$ satisfies (4), then $f \in S'$. We may assume that f is homogeneous with respect to x_1, x_2, x_3. We induct on the partial degree d of f with respect to x_1, x_2, x_3. If $d = 0$, then the assertion is trivial. Assume that $d > 0$, and put $f_j = \frac{\partial f}{\partial x_j}$, for $j = 1, 2, 3$. One sees easily that $f_j \in S$ for $j = 1, 2, 3$. By induction hypothesis, we have $f_j \in S'$ for $j = 1, 2, 3$.

Consider the complex of free S'-modules:

$$0 \to S' \overset{(\delta_1, -\delta_2, \delta_3)}{\to} S'^3 \overset{{}^tM}{\to} S'^2,$$

where ${}^tM$ is the transpose of the matrix

$$M = \begin{pmatrix} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} \end{pmatrix}$$

and δ_j is the determinant of the matrix M with j-th column deleted. Note that this is an exact sequence, since S' is flat over $K[u_1, u_2]$ and since $(\delta_1, \delta_2, \delta_3)$ is a perfect ideal of height two in $K[u_1, u_2]$. (This is Hilbert-Birch Theorem.) We have proved that the vector (f_1, f_2, f_3) is an element in S'^3 and it lies in the kernel of ${}^tM$. Thus there exists an element $C \in S'$ such that $(f_1, f_2, f_3) = C(\delta_1, -\delta_2, \delta_3)$. Since f is homogeneous with respect to x_1, x_2, x_3, we have, by Euler's identity,

$$d\, f = (f_1, f_2, f_3) \cdot (x_1, x_2, x_3) = C(\delta_1 x_1 - \delta_2 x_2 + \delta_3 x_3) = C\Delta.$$

This shows that $f \in S'$. □

Remark 7.5. Hirokazu Nasu showed that the variety $X = (\Delta = 0) \subset \mathbb{P}^4$ is isomorphic to the projection of the Segre variety $S \subset \mathbb{P}^5$ of degree three from a general point outside of S. Here $S \subset \mathbb{P}^5$ is defined as the image of the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5.$$

This is the locus of the maximal minors of a generic 2×3 matrix.

It is easy to see that any degree three homogenous polynomial $F \in K[u, v, \Delta]$ in Theorem 7.4 can be transformed into the canonical form $u^2x + uvy + v^2z$ by means of a linear transformation of the variables. This form is also known as the Macaulay dual of the trivial extension of the algebra $K[u, v]/(u, v)^3$ by the canonical module.

Remark 7.6. Let n be an arbitrary integer and let $f \in K[x_1, \ldots, x_n]$ be a homogeneous polynomial with zero Hessian. Let $\boldsymbol{h}$ be a self-vanishing system that arises from f. Assume that

$$h_j(x) \in \mathrm{Sol}(i(L(W)); R) \quad \text{for all } j. \tag{5}$$

Then by a linear change of variables, $\boldsymbol{h}$ coincides with a self-vanishing system described in Example 3.6. Namely, $h_1(x) = \cdots = h_r(x) = 0$, for some $1 \leq r < n$ and $h_j(x)$ is a function only in $x_1, \ldots, x_r$ for $j > r$.

In §6 of [6], Gordan-Noether gives a complete description of such forms f under this assumption. We have not confirmed that their assertions are correct.

We do not know any example of homogeneous polynomials with zero Hessian where (5) does not hold.

References

[1] W. Bruns and U. Vetter, *Determinantal Rings,* Lecture notes in Mathematics **1327**, Springer, 1988.

[2] C. Ciliberto and F. Russo, A. Simis, *Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian,* Adv. Math. 218 (2008), no. 6, 1759–1805.

[3] A. Alice and F. Repetto *A geometrical approach to Gordan–Noether's and Franchetta's contributions to a question posed by Hesse* Collect. Math. 60 (2009), no. 1, 27–41.

[4] O. Hesse, *Über die Bedingung, unter welcher eine homogene ganze Function von n unabhängigen Variabeln durch lineäre Substitutionen von n andern unabhängigen Variabeln auf eine homogene Function sich zurükfüren läfst, die eine Variable weniger enthält,* Journal für die reine und angewandte Mathematik, **52** (1851), 117–124.

[5] O. Hesse, *Zur Theorie der Ganzen homogenen Functionen,* Journal für die reine und angewandte Mathematik, **56** (1859), 263–269.

[6] P. Gordan and M. Noether, *Ueber die algebraischen Formen, deren Hesse'sche Determinante identisch verschwindet,* Math. Ann. **10** (1876), 547–568.

[7] T. Harima, J. Migliore, U. Nagel and J. Watanabe, *The weak and strong Lefschetz properties for Artinian K-algebras,* J. Algebra **262** (2003), 99-126.

[8] C. Lossen, *When does the Hessian determinant vanish identically?* , Bulletin of the Brazilian Mathematical Society, 35(1) (2004), 71–82.

[9] T. Maeno and J. Watanabe, *Lefschetz elements of Artinian Gorenstein algebras and Hessians of Homogeneous polynomials*, Illinois J. Mathematics, 53 (2) (2009), 591–603.

[10] M. Pasch, *Zur Theorie der Hesseschen Determinante,* Journal für die reine und angewandte Mathematik, **80** (1875), 169-176.

[11] R–M. Miró–Roig, *Determinantal ideals,* Progress in Mathematics **264**, Birkhäuser, 2008.

[12] J. Watanabe, *A remark on the Hessian of homogeneous polynomials,* in *The Curves Seminar at Queen's Volume XIII,* Queen's Papers in Pure and Appl. Math., Vol. 119, 2000, 171–178.

[13] H. Yamada, *On a theorem of Hesse — P. Gordan and M. Noether's Theory —* , Unpublished.