

# Conservative Critical Value of Certain Multivariate Tests for Normal Mean Vectors Based on Likelihood Ratio Test with Unknown Covariance Matrix

by

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## Abstract

We discuss conservative critical values of a certain multivariate one-sided test and a certain multivariate two-sided test for normal mean vectors based on the likelihood ratio test under the assumption that the covariance matrix is unknown. The conservative critical value of the former test is obtained by an upper bound of the distribution of the likelihood ratio test statistic for all covariance matrices derived by Perlman (1969) [4]. For the latter test we construct an upper bound of the distribution of the likelihood ratio test statistic by referring to Perlman (1969) [4] intended to obtain a conservative critical value. We investigate the conservation for two tests through numerical examples regarding Type I error.

**Key words:**  $F$ -distribution, inequality, Type I error

## 1 Introduction

Multivariate one-sided tests and multivariate two-sided tests for normal mean vectors which are multivariate analogs of the one-sided test and the two-sided test for normal means have been studied by many researchers. Kudo (1963) [3] discussed a multivariate one-sided test for testing whether a normal mean vector having nonnegative components is equal to zero or not based on the likelihood ratio test under the assumption that the covariance matrix is known. Kudo (1963) [3] derived the distribution of the likelihood ratio test statistic under the null hypothesis to determine a critical value for a specified significance level. However, it is difficult to derive the distribution of the likelihood ratio test statistic under the assumption that the covariance matrix is unknown. Perlman (1969) [4] derived an upper bound of the distribution of the likelihood ratio test statistic for all covariance matrices under the null hypothesis. We can obtain a conservative critical value for a specified significance level by using the upper bound.

On the other hand, it is also difficult to derive the distribution of the likelihood ratio test statistic of the multivariate two-sided test for testing whether a normal mean vector is equal to zero or not under the assumption that all components of the normal mean vector are simultaneously nonnegative or nonpositive and the covariance matrix is unknown. In this study we construct an upper bound of the distribution of the likelihood ratio test statistic of the multivariate two-sided test under the null hypothesis by applying Perlman (1969) [4]’s idea. Then we determine a conservative critical value based on the upper bound. We investigate the conservations for two tests through numerical examples regarding Type I error.

## 2 Multivariate one-sided test

In this Section we discuss a multivariate one-sided test for testing whether a normal mean vector having nonnegative components is equal to zero or not based on the likelihood ratio test under the assumption that the covariance matrix is unknown. First we discuss the process for deriving the likelihood ratio test statistic. Then we discuss an upper bound of the distribution of the likelihood ratio test statistic for all covariance matrices under the null hypothesis derived by Perlman (1969) [4].

## 2.1 Setting up hypotheses

Assume  $p$ -dimensional random variable vector  $\mathbf{X}$  is distributed according to multivariate normal  $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . Here a mean vector  $\boldsymbol{\mu}$  and a covariance matrix  $\boldsymbol{\Sigma}$  are unknown. Letting  $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_p)'$ , assume

$$\mu_1 \geq 0, \mu_2 \geq 0, \dots, \mu_p \geq 0. \quad (2.1)$$

If  $\mu_i > 0$  for at least one  $i (= 1, 2, \dots, p)$  in (2.1), we express it as  $\boldsymbol{\mu} \geq^* \mathbf{0}$ . We set up a null hypothesis  $H_0$  versus its alternative hypothesis  $H_1$  as

$$H_0 : \boldsymbol{\mu} = \mathbf{0} \quad \text{vs.} \quad H_1 : \boldsymbol{\mu} \geq^* \mathbf{0}. \quad (2.2)$$

We test (2.2) by using a sample  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  from  $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ .

## 2.2 Likelihood ratio test statistic

We summarize the process for deriving likelihood ratio test statistic for (2.2). Letting  $L$  be the likelihood function,

$$\begin{aligned} L &= \frac{1}{(2\pi)^{\frac{np}{2}} |\boldsymbol{\Sigma}|^{\frac{n}{2}}} \exp \left[ -\frac{1}{2} \left\{ \sum_{i=1}^n (\mathbf{X}_i - \bar{\mathbf{X}})' \boldsymbol{\Sigma}^{-1} (\mathbf{X}_i - \bar{\mathbf{X}}) + n(\bar{\mathbf{X}} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu}) \right\} \right] \\ &= \frac{1}{(2\pi)^{\frac{np}{2}} |\boldsymbol{\Sigma}|^{\frac{n}{2}}} \exp \left[ -\frac{1}{2} \text{tr} \boldsymbol{\Sigma}^{-1} \{ \mathbf{S} + n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' \} \right] \end{aligned}$$

where

$$\bar{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i, \quad \mathbf{S} = \sum_{i=1}^n (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})'$$

Since

$$\log L = -\frac{np}{2} \log 2\pi - \frac{n}{2} \log |\boldsymbol{\Sigma}| - \frac{1}{2} \text{tr} \boldsymbol{\Sigma}^{-1} \{ \mathbf{S} + n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' \}, \quad (2.3)$$

the value of  $\boldsymbol{\Sigma}$  maximizing  $L$  is given as

$$\boldsymbol{\Sigma} = \frac{1}{n} \{ \mathbf{S} + n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' \} \quad (2.4)$$

by Anderson (2003) [1] Lemma 3.2.2. By substituting (2.4) into (2.3) we obtain

$$\log L = -\frac{np}{2} (\log 2\pi + 1) - \frac{n}{2} \log \left| \frac{1}{n} \{ \mathbf{S} + n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' \} \right|.$$

Since

$$|\mathbf{S} + n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})'| = |\mathbf{S}| \{1 + n(\bar{\mathbf{X}} - \boldsymbol{\mu})' \mathbf{S}^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu})\}$$

by Anderson (2003) [1] Corollary A.3.1, we obtain

$$\log L = -\frac{np}{2}(\log 2\pi + 1) - \frac{n}{2} \log \left| \frac{\mathbf{S}}{n} \right| - \frac{n}{2} \log \{1 + n(\bar{\mathbf{X}} - \boldsymbol{\mu})' \mathbf{S}^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu})\}. \quad (2.5)$$

The maximum likelihood estimator of  $\boldsymbol{\mu}$  under  $H_0$  is equal to  $\mathbf{0}$ . By substituting  $\boldsymbol{\mu} = \mathbf{0}$  into (2.5) we obtain

$$\log L = -\frac{np}{2}(\log 2\pi + 1) - \frac{n}{2} \log \left| \frac{\mathbf{S}}{n} \right| - \frac{n}{2} \log(1 + n\bar{\mathbf{X}}' \mathbf{S}^{-1} \bar{\mathbf{X}}). \quad (2.6)$$

The maximum likelihood estimator of  $\boldsymbol{\mu}$  under  $H_0 \cup H_1$  is determined as the value of  $\boldsymbol{\mu}$  which minimizes  $(\bar{\mathbf{X}} - \boldsymbol{\mu})' \mathbf{S}^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu})$ . Therefore, it is the orthogonal projection  $\pi_{\mathbf{S}}(\bar{\mathbf{X}}, \boldsymbol{\Theta})$  of  $\bar{\mathbf{X}}$  onto

$$\boldsymbol{\Theta}_1 = \{\mathbf{x} = (x_1, x_2, \dots, x_p)' \in \mathbf{R}^p | x_1 \geq 0, x_2 \geq 0, \dots, x_p \geq 0\}$$

with regard to Mahalanobis distance  $\|\cdot\|_{\mathbf{S}}$  based on  $\mathbf{S}$ . Specifically,  $\pi_{\mathbf{S}}(\bar{\mathbf{X}}, \boldsymbol{\Theta}_1)$  is the nearest point in  $\boldsymbol{\Theta}_1$  from  $\bar{\mathbf{X}}$ . In this study we use well-known properties of orthogonal projections without any proofs. For example, they are explained in Kudo (1963) [3] and Perlman (1969) [4]. Letting  $\boldsymbol{\mu} = \pi_{\mathbf{S}}(\bar{\mathbf{X}}, \boldsymbol{\Theta}_1)$  in (2.5), we obtain

$$\log L = -\frac{np}{2}(\log 2\pi + 1) - \frac{n}{2} \log \left| \frac{\mathbf{S}}{n} \right| - \frac{n}{2} \log \{1 + n \|\bar{\mathbf{X}} - \pi_{\mathbf{S}}(\bar{\mathbf{X}}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2\}. \quad (2.7)$$

By (2.6) and (2.7) the likelihood ratio for (2.2) is given by

$$\left( \frac{1 + n \|\bar{\mathbf{X}} - \pi_{\mathbf{S}}(\bar{\mathbf{X}}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2}{1 + n \|\bar{\mathbf{X}}\|_{\mathbf{S}}^2} \right)^{\frac{n}{2}}.$$

Letting  $\mathbf{Z} = \sqrt{n}\bar{\mathbf{X}}$ , we obtain

$$n \|\bar{\mathbf{X}}\|_{\mathbf{S}}^2 = \|\mathbf{Z}\|_{\mathbf{S}}^2, \quad n \|\bar{\mathbf{X}} - \pi_{\mathbf{S}}(\bar{\mathbf{X}}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2 = \|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2.$$

Then

$$\left( \frac{1 + n \|\bar{\mathbf{X}} - \pi_{\mathbf{S}}(\bar{\mathbf{X}}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2}{1 + n \|\bar{\mathbf{X}}\|_{\mathbf{S}}^2} \right)^{\frac{n}{2}} = \left( \frac{1 + \|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2}{1 + \|\mathbf{Z}\|_{\mathbf{S}}^2} \right)^{\frac{n}{2}}.$$

Here

$$\|\mathbf{Z}\|_{\mathbf{S}}^2 = \|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2 + \|\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2.$$

Therefore, by the likelihood ratio test criteria for (2.2) we reject  $H_0$  when

$$\bar{F}_u = \frac{\|\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2}{1 + \|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}^2} > c$$

for a specified critical value  $c$ .

### 2.3 Upper bound of the distribution of $\bar{F}_u$

The critical value  $c$  should be determined so that

$$P(\bar{F}_u > c) = \alpha$$

under  $H_0$  for a specified significance level  $\alpha$ . However, it is difficult to determine the distribution of  $\bar{F}_u$  under  $H_0$ . Perlman (1969) [4] derived the upper bound of the distribution of  $\bar{F}_u$  for all covariance matrices under  $H_0$ . The bound is expressed by using  $F$ -distributions. Specifically,

$$\max_{\boldsymbol{\Sigma}} P(\bar{F}_u > c) = \frac{1}{2}P(((p-1)/(n-p))F_{p-1, n-p} > c) + \frac{1}{2}P((p/(n-p))F_{p, n-p} > c). \quad (2.8)$$

If we determine  $c$  so that

$$\frac{1}{2}P(((p-1)/(n-p))F_{p-1, n-p} > c) + \frac{1}{2}P((p/(n-p))F_{p, n-p} > c) = \alpha, \quad (2.9)$$

we obtain

$$P(\bar{F}_u > c) \leq \alpha,$$

which shows  $c$  is a conservative critical value.

## 3 Multivariate two-sided test

Next, we discuss a multivariate two-sided test for testing whether the normal mean vector is equal to zero or not under the assumption that all components of a normal mean vector are simultaneously nonnegative or nonpositive and the covariance matrix is unknown. We construct an upper bound of the distribution of the likelihood ratio test statistic of the multivariate two-sided test under  $H_0$  by applying Perlman (1969) [4]'s idea. Then we determine a conservative critical value based on the upper bound.

### 3.1 Setting up hypotheses

Assume  $p$ -dimensional random variable vector  $\mathbf{X}$  is distributed according to multivariate

normal  $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . Letting  $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_p)'$ , assume

$$\mu_1 \geq 0, \mu_2 \geq 0, \dots, \mu_p \geq 0$$

or

$$\mu_1 \leq 0, \mu_2 \leq 0, \dots, \mu_p \leq 0. \quad (3.1)$$

If  $\mu_i < 0$  for at least one  $i (= 1, 2, \dots, p)$  in (3.1), we express it as  $\boldsymbol{\mu} \leq^* \mathbf{0}$ . We set up a null hypothesis  $H_0$  versus its alternative hypothesis  $H_1$  as

$$H_0 : \boldsymbol{\mu} = \mathbf{0} \text{ vs. } H_1 : \boldsymbol{\mu} \geq^* \mathbf{0} \text{ or } \boldsymbol{\mu} \leq^* \mathbf{0}. \quad (3.2)$$

We call the test (3.2) a multivariate two-sided test. We test (3.2) by using a sample  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  from  $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ .

### 3.2 Likelihood ratio test statistic

The likelihood ratio test statistic for (3.2) is given by

$$\bar{\bar{F}} = \frac{\|\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})\|_{\mathbf{S}}^2}{1 + \|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})\|_{\mathbf{S}}^2}$$

through a process similar to that for determining  $\bar{F}_u$ . Here

$$\boldsymbol{\Theta} = \{\mathbf{x} = (x_1, x_2, \dots, x_p)' \in \mathbf{R}^p \mid x_1 \geq 0, x_2 \geq 0, \dots, x_p \geq 0 \text{ or } x_1 \leq 0, x_2 \leq 0, \dots, x_p \leq 0\}.$$

Specifically, we reject  $H_0$  when  $\bar{\bar{F}} > c$  for a specified critical value  $c$ . Letting

$$\boldsymbol{\Theta}_2 = \{\mathbf{x} = (x_1, x_2, \dots, x_p)' \in \mathbf{R}^p \mid x_1 \leq 0, x_2 \leq 0, \dots, x_p \leq 0\},$$

$\boldsymbol{\Theta} = \boldsymbol{\Theta}_1 \cup \boldsymbol{\Theta}_2$  and

$$\|\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})\|_{\mathbf{S}} = \max\{\|\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}, \|\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_2)\|_{\mathbf{S}}\},$$

$$\|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta})\|_{\mathbf{S}} = \min\{\|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_1)\|_{\mathbf{S}}, \|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_2)\|_{\mathbf{S}}\}.$$

Therefore, letting

$$\bar{F}_l = \frac{\|\pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_2)\|_{\mathbf{S}}^2}{1 + \|\mathbf{Z} - \pi_{\mathbf{S}}(\mathbf{Z}, \boldsymbol{\Theta}_2)\|_{\mathbf{S}}^2},$$

we obtain

$$\bar{\bar{F}} = \max\{\bar{F}_u, \bar{F}_l\}. \quad (3.3)$$

Note that  $\bar{F}_l$  is the likelihood ratio test statistic for the multivariate lower-sided test

$$H_0 : \boldsymbol{\mu} = \mathbf{0} \quad \text{vs.} \quad H_1 : \boldsymbol{\mu} \leq^* \mathbf{0}.$$

### 3.3 Upper bound of the distribution of $\bar{\bar{F}}$

The critical value  $c$  should be determined so that

$$P(\bar{\bar{F}} > c) = \alpha$$

under  $H_0$  for a specified significance level  $\alpha$ . However, it is difficult to derive the distribution of  $\bar{\bar{F}}$ . By (3.3)

$$P(\bar{\bar{F}} > c) \leq P(\bar{F}_u > c) + P(\bar{F}_l > c).$$

Since  $P(\bar{F}_u > c) = P(\bar{F}_l > c)$  under  $H_0$ , we obtain

$$P(\bar{\bar{F}} > c) \leq 2P(\bar{F}_u > c). \quad (3.4)$$

By (2.8) and (3.4)

$$\max_{\Sigma} P(\bar{\bar{F}} > c) \leq P(((p-1)/(n-p))F_{p-1, n-p} > c) + P(((p/(n-p))F_{p, n-p} > c).$$

If we determine  $c$  so that

$$P(((p-1)/(n-p))F_{p-1, n-p} > c) + P(((p/(n-p))F_{p, n-p} > c) = \alpha, \quad (3.5)$$

we obtain

$$P(\bar{\bar{F}} > c) \leq \alpha,$$

which shows  $c$  is a conservative critical value for (3.2).

## 4 Numerical examples

In Section 2 we discussed Perlman (1969) [4]'s upper bound of the distribution of the likelihood ratio test statistic of the multivariate one-sided test under the null hypothesis. In Section 3 we constructed an upper bound of the distribution of the likelihood ratio test statistic of the multivariate two-sided test under the null hypothesis by applying Perlman (1969) [4]'s idea. Although we can obtain conservative critical values by using the bounds

Table 1: Type I error of multivariate one-sided test by using conservative critical values for  $p = 2$ 

| $\rho$   | -0.9   | -0.6   | -0.3   | 0      | 0.3    | 0.6    | 0.9    |
|----------|--------|--------|--------|--------|--------|--------|--------|
| $n = 20$ | 0.0170 | 0.0234 | 0.0274 | 0.0307 | 0.0353 | 0.0387 | 0.0437 |
| $n = 30$ | 0.0171 | 0.0226 | 0.0276 | 0.0308 | 0.0344 | 0.0385 | 0.0441 |

Table 2: Type I error of multivariate two-sided test by using conservative critical values for  $p = 2$ 

| $\rho$   | -0.9   | -0.6   | -0.3   | 0      | 0.3    | 0.6    | 0.9    |
|----------|--------|--------|--------|--------|--------|--------|--------|
| $n = 20$ | 0.0170 | 0.0217 | 0.0268 | 0.0310 | 0.0339 | 0.0358 | 0.0372 |
| $n = 30$ | 0.0170 | 0.0222 | 0.0267 | 0.0307 | 0.0337 | 0.0361 | 0.0384 |

for two tests, we should investigate the conservation by using numerical examples.

First, let  $p = 2$ . Let  $\alpha = 0.05$  and  $n = 20, 30$ . By (2.9) we obtain conservative critical values for multivariate one-sided test  $c = 0.334, 0.203$  for  $n = 20, 30$ , respectively. By (3.5) we obtain conservative critical values for multivariate two-sided test  $c = 0.437, 0.262$  for  $n = 20, 30$ , respectively. We calculate Type I error of two tests for specified  $\Sigma$ . Specifically,

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

where  $\rho = -0.9, -0.6, -0.3, 0, 0.3, 0.6, 0.9$ . Tables 1 and 2 give Type I error of two tests by using conservative critical values. We obtain the probability by Monte Carlo simulations with 100,000 random numbers. Tables show that the critical value is less conservative as  $\rho$  is larger from  $-0.9$  to  $0.9$  for two tests. The critical value of multivariate two-sided test is uniformly more conservative compared to that of multivariate one-sided test. The difference of Type I error between two tests is larger as  $\rho$  is larger from  $-0.9$  to  $0.9$  for two tests. Type I error does not vary remarkably as the sample size  $n$  varies from 20 to 30 for two tests.

Next, let  $p = 3$ . Let  $\alpha = 0.05$  and  $n = 20, 30$ . By (2.9) we obtain conservative critical values for multivariate one-sided test  $c = 0.503, 0.295$  for  $n = 20, 30$ , respectively. By (3.5) we obtain  $c = 0.638, 0.367$  for  $n = 20, 30$ , respectively. Tables 3 and 4 give Type I error for two tests by using conservative critical values for

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$$

with  $\rho = -0.3, 0, 0.3, 0.6, 0.9$ . We obtain the probability by Monte Carlo simulations. The results are similar to those for  $p = 2$ .

Table 3: Type I error of multivariate one-sided test by using conservative critical values for  $p = 3$

| $\rho$   | -0.3   | 0      | 0.3    | 0.6    | 0.9    |
|----------|--------|--------|--------|--------|--------|
| $n = 20$ | 0.0139 | 0.0215 | 0.0271 | 0.0342 | 0.0435 |
| $n = 30$ | 0.0140 | 0.0212 | 0.0270 | 0.0348 | 0.0421 |

Table 4: Type I error of multivariate two-sided test by using conservative critical values for  $p = 3$

| $\rho$   | -0.3   | 0      | 0.3    | 0.6    | 0.9    |
|----------|--------|--------|--------|--------|--------|
| $n = 20$ | 0.0141 | 0.0199 | 0.0275 | 0.0322 | 0.0339 |
| $n = 30$ | 0.0133 | 0.0206 | 0.0258 | 0.0317 | 0.0348 |

## 5 Conclusions

In this study we have discussed the conservative critical values derived by the upper bounds of the likelihood ratio test statistics for the multivariate one-sided test and the multivariate two-sided test under the assumption that the covariance matrix is unknown. We gave some numerical examples regarding Type I error by using conservative critical values intended to investigate the conservation. Although our conservative critical value can be determined easily, we confirmed that it is occasionally fairly conservative through numerical examples.

On the other hand, Fraser, Guttman and Srivastava (1991) [2] constructed another kind of critical value of the multivariate one-sided test based on the likelihood ratio test. Their critical value satisfies a specified significance level exactly under the condition that the

orthogonal projection is determined. It will be possible to construct a conservative critical value of the multivariate two-sided test by referring to their idea. We will compare two kinds of conservative critical values through numerical examples regarding Type I error in the future. Furthermore, in this study we have not discussed the power of the test. We should calculate the power of our tests under various situations.

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