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On the Minimal Number of Generators of the Quotient of a Complete Intersection by a General Linear Form

by

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Abstract

Let k be a field of characteristic 0, and let $I \subset k[x_1, \cdots, x_r]$ be a complete intersection minimally generated by r homogeneous elements of degrees $d_1, \cdots, d_r$ with $2 \leq d_1 \leq \cdots \leq d_r$. Then the quotient of I by a general linear form is minimally generated by r elements if and only if $d_r \leq d_1 + \cdots + d_{r-1} - (r-1)$.

1 Introduction

Let $I = (f_1, \cdots, f_r)$ be an ideal of height r, generated by homogeneous elements in a polynomial ring in r variables over a field of characteristic 0. When they are reduced by a general linear form ℓ, we have either $\mu(I + (\ell)/(\ell)) = r$ or $r - 1$. (Throughout the paper, μ denotes the minimal number of generators.) The purpose of this paper is to prove that the number

$$\mu(I + (\ell)/(\ell))$$

depends only on the degrees of the generators. (For details see Theorem 6 in §3.)

One sees easily that the above statement is true for monomial complete intersections. Then it will be conceived that it is true for almost all complete intersections with possibly

a few exceptional cases depending on certain algebraic relations of the coefficients of the generators. On the other hand one should also notice that it is a highly non-trivial fact that the statement is true for *all* complete intersections.

If we assume $r = 3$, the main theorem of this paper was proved in two quite different ways, in [1] and in [7].

In [1] it was proved as a corollary of a much stronger result which says that standard graded Artinian rings of embedding codimension three have the weak Lefschetz property. In the earlier paper [7], the method was to replace the coefficient field k by a discrete valuation ring $k[t]_{(t)}$, and make use of the parameter of the DVR to show that the reduction of the ideal by a general linear form does not lose much information of the ideal (as long as the number of generator is concerned). In an attempt to generalize this method to an arbitrary r, difficulty arises because the coefficients in the equation (2) in §3 are not unique. In this paper we show that the same method works for any $r \geq 3$ in spite of the non-uniqueness of the coefficients.

The basic idea of the proof is to show that the non-uniqueness of the coefficients can be canceled by the fact that any syzygy of the reduced polynomials lift to a syzygy of the original polynomials. This is done in Lemma 4. It will turn out that the expression in the equation is "essentially unique." In this sense our proof is a generalization of [7]. The author tried to make the proof readable independent of [7] as much as possible. Thus some overlappings were not avoided. In §4 we show some examples of compete intersections which have the weak Lefschetz property as an immediate consequence of Theorem 6.

After the present paper was finished the paper [2] came to the author's attention. It should be noted that the main result of this paper (Theorem 6) can be proved by applying [2, Proposition 3.11], which they call the socle lemma. The socle lemma itself is a consequence of a series of more complicated propositions of the same paper, whose proof is by no means easy. Our proof is completely different from theirs, and the result of this paper does not seem to be stated explicitly in [2] or anywhere else.

2 Some lemmas

We start with a brief discussion of a general element. Let $Q = \bigoplus_{i \geq 0} Q_i$ be a graded k-algebra, where $k = Q_0$ is a field and $Q = k[Q_1]$. Suppose that Q_1 is spanned by $x_1, x_2, \cdots, x_r$. Let $\xi_1, \cdots, \xi_r$ be indeterminates. Let $k^* = k(\xi_1, \cdots, \xi_r)$ be the rational

function field and put $Q^* = Q \otimes_k k^*$. Then we call the element $L = \xi_1 x_1 + \cdots + \xi_r x_r \in Q^*$ a generic linear form of Q. Let $I \subset Q$ be an ideal. We say that $\ell \in Q$ is a general linear form for I if it plays the same role for I as the generic linear form for IQ^*, depending on a particular situation for various purposes. In most cases a general linear form exists and they exist dense for a given ideal or ideals. (cf. [5, Appendix].) Also in most cases there is no harm in replacing the ideal I by IQ^* and working over the ring Q^*. For these reasons we treat a generic linear form as though it were an element of Q, unless otherwise specified.

For the rest of this section we prove some lemmas necessary for a proof of Main Theorem 6 in §3.

Lemma 1 *Let $R = \oplus_{i \geq 0} R_i$ be a graded ring over a local ring R_0 and let M denote the unique homogeneous maximal ideal of R. Let $I \subset R$ be a homogeneous ideal. Suppose that $\ell \in M$ is a non-zero-divisor of R/I. Then $\mu(I + (\ell)/(\ell)) = \mu(I)$.*

Proof. See proof of [7, Lemma 5].

Lemma 2 *Let k be a field of characteristic 0 and $R = k[x_1, \cdots, x_r]$ the polynomial ring over k. Suppose that a subset $S \subset R_1$ consists linear forms as many as* $\dim R_d$*, in which any n forms are linearly independent. Then the set $\{f^d | f \in S\}$ spans R_d.*

Proof. See Iarrobino [3, Proposition 3.2].

Lemma 3 *Let R_0 be an integral domain which contains a field of characteristic 0. Let $R = R_0[z]$ be the polynomial ring in a variable z over R_0. Let $\hat{R} = R_0[\alpha]$, where α is a new variable. Let $0 \neq g \in R_0$. Define the homomorphism of R_0-algebras $\phi : R \to \hat{R}$ by $z \mapsto g\alpha$. Let $A(\alpha) \in \hat{R}$. Then the following conditions are equivalent.*

(i) *$A(\alpha) = \phi(f)$ for some $f \in R$.*

(ii) *$\dfrac{d^n}{d\alpha^n} A(0)$ is divisible by g^n for all $n = 1, 2, \cdots$.*

Proof. For $f \in R = R_0[z]$ let us write $f = f(z)$. It is a polynomial in the variable z with coefficients in R_0. Assume (i), so $A(\alpha) = f(g\alpha)$. Then $\frac{d^n}{d\alpha^n} A(\alpha) = f^{(n)}(g\alpha) g^n$, where $f^{(n)}(z) = \frac{d^n}{dz^n} f(z)$. Hence (ii) follows.

To prove the other implication write $A(\alpha) = \sum_{n \geq 0} A_n \alpha^n$, which is the Taylor expansion of $A(\alpha)$ in powers of α. (This means that $A_n = (1/n!) A^{(n)}(0)$.) Note that it is a finite sum because we are dealing with a polynomial. Assuming that (ii) is true put $f = f(z) = \sum_{n \geq 0} (A_n / g^n) z^n$, which is an element in R. Then $f(g\alpha) = A(\alpha)$ as desired.

Lemma 4 *Let R_0 be a local ring (or a graded ring) containing a field of characteristic 0, and $R = R_0[[t]]$ the formal power series ring over R_0 in an indeterminate. For $f(t) \in R$ we denote by $f(0)$ the constant term of $f(t)$. Let $f_i(t), i = 1, 2, \cdots, r$ be elements of R. Suppose that the elements $f_1(0), \cdots, f_r(0)$ form a regular sequence of R_0. If R_0 is a local (resp. graded) ring then we assume that they are non-unit elements (resp. homogeneous elements of positive degree). Let $s = (s_1, \cdots, s_r)$ be a syzygy of $(f_1(0), \cdots, f_r(0))$ over R_0, i.e, $\sum_{i=1}^{r} s_i f_i(0) = 0$ with $s_i \in R_0$. Then for $n \geq 0$ given, there is a syzygy $S = (S_1, \cdots, S_r)$ of $(f_1(t), \cdots, f_r(t))$ over R with the following properties.*

1. $\frac{d^n}{dt^n} S_i(0) = s_i$, *for* $i = 1, \cdots, r$,
2. $\frac{d^m}{dt^m} S_i(0) = 0, \forall m < n$, *for* $i = 1, \cdots, r$.

Proof. Let s_{ij} be the Koszul relation of $(f_i(0))$ which has $f_j(0)$ at the i-th place, $-f_i(0)$ at the j-th place and 0's at the other places. It is well known that any syzygy of a regular sequence over a local ring or a graded ring is a linear combination of such syzygies. Suppose that $s = \sum c_{ij} s_{ij}$ with $c_{ij} \in R_0$ is given. Let S_{ij} be the lift of s_{ij}, i.e., $S_{ij} = (0, \cdots, f_j(t), \cdots, -f_i(t), \cdots, 0)$. Then

$$S = (1/n!) t^n \sum c_{ij} S_{ij}$$

is a desired syzygy.

Lemma 5 *Let $(R, \mathfrak{m}, k)$ be an Artinian Gorenstein graded ring such that $R = k[R_1]$. Let d be the socle degree of R. Then we have $0 : \mathfrak{m}^i = \mathfrak{m}^{d+1-i}$.*

Proof. See [6, Proposition 9][1].

3 Theorem and proof

Theorem 6 *Let $R = k[x_1, \cdots, x_r]$ be the polynomial ring over a field k of characteristic 0. Let I be a complete intersection ideal of R generated by homogeneous elements $f_1, \cdots, f_r \in R$ of degrees $d_1, \cdots, d_r$ respectively, where we assume that $2 \leq d_1 \leq d_2 \leq \cdots \leq d_r$. Then the following conditions are equivalent.*

[1] In the statement of [6, Proposition 9] there is a misprint. The assumption that $(A, \mathfrak{m}, k)$ is an "Artin local ring" should have been "Artin Gorenstein local ring." The proof is carried out under this assumption. The author would like to borrow this place to correct it.

(i) $\mu(I + \ell R/\ell R) = r$, *for a general linear form* $\ell \in R$.

(ii) $d_r \leq d_1 + d_2 + \cdots + d_{r-1} - (r-1)$.

Remark 7 Assume that $d_r - d_i < d_j$, for all $i, j < r$ such that $i \neq j$. (e.g., $d_1 = d_2 = \cdots = d_r$.) Then the proof of [7] works verbatim, because the coefficients in the equation (2) below are unique.

Proof of Theorem 6. (i) $\Rightarrow$ (ii). Let ℓ be a general linear form of R. Then $f_1, \cdots, f_{r-1}, \ell$ is a regular sequence. The socle degree of $R/(f_1, f_2, \cdots, f_{r-1}, \ell)$ is $d_1 + \cdots + d_{r-1} - (r-1)$. Hence if (ii) is not true, then f_r is contained in the ideal $(f_1, \cdots, f_{r-1})R \bmod \ell R$, contradicting (i).

(ii)$\Rightarrow$(i). We may apply a general linear transformation of the variables, so we assume that x_r is a generic variable. Let α be a new variable with $\deg \alpha = 0$ and put $\hat{R} = k[\alpha][x_1, x_2, \cdots, x_{r-1}]$, which is the polynomial ring in $x_1, \cdots, x_{r-1}$ with coefficients in $k[\alpha]$. Let g be a general linear combination of $x_1, x_2, \ldots, x_{r-1}$ with coefficients in k (so g is a general linear form of $k[x_1, \cdots, x_{r-1}]$) and consider the map $\phi : R \to \hat{R}$ defined by

$$\begin{cases} x_i \mapsto x_i, \text{ for } i = 1, 2 \cdots, r-1 \ , \\ x_r \mapsto g\alpha. \end{cases}$$

Note that ϕ is a graded homomorphism. Put $\hat{f}_i = \phi(f_i)$, $\hat{I} = (\hat{f}_1, \cdots, \hat{f}_r)\hat{R}$. We will write $f_i = f_i(x_r) \in k[x_1 \cdots, x_{r-1}][x_r]$ for $i = 1, \cdots, r$ and regard them as polynomials in the variable x_r with coefficients in $k[x_1, \cdots, x_{r-1}]$. Then

$$\hat{f}_i = f_i(g\alpha), \quad i = 1, 2, \cdots, r.$$

By way of contradiction we assume that (i) of Theorem 6 is not true. Our first objective is to derive the inclusion

$$f_r(g\alpha) \in (f_1(g\alpha), \cdots, f_{r-1}(g\alpha)) \tag{1}$$

in the ring $k[\alpha]_{(\alpha)}[x_1, \cdots, x_{r-1}]$, which is the polynomial ring in $x_1, \cdots, x_{r-1}$ over the discrete valuation ring $k[\alpha]_{(\alpha)}$. The failure of (i) means that

$$\mu(I + (\ell)/(\ell)) = r - 1$$

for all general linear form ℓ. Notice that, for any $\alpha_0 \in k$, we have the isomorphism

$$\hat{R}/\hat{I} + (\alpha - \alpha_0)\hat{R} \cong R/(I + (\alpha_0 g - x_r))R.$$

The element $\alpha_0 g - x_r$ is a general linear form for R/I for all but a finite number of values of α_0, and particularly it is general for $\alpha_0 = 0$ since x_r is a generic variable. It follows that the map $k[\alpha] \to \hat{R}/\hat{I}$ is locally flat at $\alpha = 0$ because the two fibers have the same dimension (which is $d_1 d_2 \cdots d_{r-1}$). Thus α is a non-zero-divisor for $\mathcal{I} := \hat{I}\hat{R} \otimes_{k[\alpha]} k[\alpha]_{(\alpha)}$. Hence it is possible to apply Lemma 1 to conclude:

$$\mu(\mathcal{I}) = r - 1.$$

This means that the element $f_r(g\alpha)$, or any one of the elements of the greatest degree, is expressible as a linear combination of the others. Thus we have derived the inclusion (1) from the assumption that the statement (i) of Theorem 6 is not true. Now we write:

$$f_r(g\alpha) = \sum_{i=1}^{r-1} A_i(\alpha) f_i(g\alpha), \tag{2}$$

where $A_i(\alpha) \in k[\alpha]_{(\alpha)}[x_1, \cdots, x_{r-1}]$ are homogeneous elements. (Recall that deg $\alpha = 0$.) The notation $A_i(\alpha)$ is used to mean that they are treated as functions in α. Note that the ring $k[\alpha]_{(\alpha)}[x_1, \cdots, x_{r-1}]$ is a subring of $k[x_1, \cdots, x_{r-1}][[\alpha]]$. Let $A_i^{(n)}(\alpha)$ denotes the n-th derivative of $A_i(\alpha)$ with respect to α, and $A_i^{(n)}(0)$ the n-th derivative of $A_i(\alpha)$ followed by the reduction by α. We claim the following

CLAIM. It is possible to choose the equation (2) in which the coefficients $A_i(\alpha)$'s have the property that each $A_i^{(n)}(0)$ is divisible by g^n for all $n \geq 0$.

Once this is proved then as in the proof of Lemma 3, it is possible to make a substitution $\alpha = x_r/g$ in the equation (2). It will give us the inclusion

$$f_r(x_r) \in (f_1(x_r), \cdots, f_{r-1}(x_r))R$$

which contradicts the assumption that $\mathrm{ht}(f_1, \cdots, f_r) = r$, and it will complete the proof of Theorem 6.

Proof of Claim is done by induction on n in two steps, Lemma 9 and Lemma 10 below. Before we proceed to these lemmas, we make a definition.

Definition 8 *We say that* $(A_i^n(0)/g^n)$*'s in the equation* (2) *do not depend on* g *if for any general linear form* $h \in k[x_1, \cdots, x_{r-1}]$ *there is an equation* (3) *as in Lemma 9 such that* $A_i^{(n)}(0)/g^n = B_i^{(n)}(0)/h^n$*, for any* $i = 1, 2, \cdots, r-1$.

Lemma 9 *Let* $h \in k[x_1, \cdots, x_{r-1}]$ *be any general linear form like* g*. Then it is possible to have an expression like the equation* (2)

$$f_r(h\alpha) = \sum_{i=1}^{r-1} B_i(\alpha) f_i(h\alpha), \tag{3}$$

such that $B_i(0) = A_i(0)$ for $i = 1, 2, \cdots, r-1$.

Proof. Anyway there is an expression like (3) for the same reason as for g. Since $f_r(0) = \sum A_i(0) f_i(0) = \sum B_i(0) f_i(0)$, the coefficients $(A_i(0))$ and $(B_i(0))$ are different only by a syzygy of $(f_i(0))$. Regard the ring $k[\alpha]_{(\alpha)}[x_1, \cdots, x_{r-1}]$ as a subring of $k[x_1, \cdots, x_{r-1}][[\alpha]]$. Since the elements $f_1(0), f_2(0), \cdots, f_{r-1}(0)$ form a regular sequence, we may apply Lemma 4 with $n = 0$. Thus it is possible to modify the coefficients $(B_1(h\alpha), \cdots, B_{r-1}(h\alpha))$ by adding a syzygy so that their reduction by α coincide with $(A_i(0))$.

Lemma 10 (Key Lemma) *Assume the following.*

(I) *For any* m *such that* $0 \leq m < n$, $A_i^{(m)}(0)$ *is divisible by* g^m*, for* $i = 1, 2, \cdots, r-1$.

(II) *For any* m *such that* $0 \leq m < n$, $A_i^{(m)}(0)/g^m$ *does not depend on* g*, for* $i = 1, 2, \cdots, r-1$. (See Definition 8.)

Then there exist elements $\tilde{A}_i(\alpha) \in k[\alpha]_{(\alpha)}[x_1, \cdots, x_{r-1}]$ *with the following properties.*

(i) $f_r(g\alpha) = \sum_{i=1}^{r-1} \tilde{A}_i(\alpha) f_i(g\alpha)$,

(ii) $\tilde{A}_i^{(m)}(0) = A_i^{(m)}(0)$, $\forall m < n$, *for* $i = 1, 2, \cdots, r-1$.

(iii) $\tilde{A}_i^{(n)}(0)$ *is divisible by* g^n*, for* $i = 1, 2, \cdots, r-1$.

(iv) $(\tilde{A}_i^{(n)}(0)/g^n)$ *does not depend on* g*, for* $i = 1, 2, \cdots, r-1$.

Proof of Lemma 10. Differentiate the equation (2) n times with respect to α. We obtain:

$$f_r^{(n)}(g\alpha) g^n = \sum_{i=1}^{r-1} \sum_{j=0}^{n} \binom{n}{j} A_i^{(n-j)}(\alpha) f_i^{(j)}(g\alpha) g^j \tag{4}$$

(We are writing $f_i'(x_r) = \dfrac{d}{dx_r} f_i(x_r)$, and $A_i'(\alpha) = \dfrac{d}{d\alpha} A_i(\alpha)$.) In the equation (4) set $\alpha = 0$, and rewrite it as

$$f_r^{(n)}(0)g^n - \sum_{i=1}^{r-1}\sum_{j=1}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) A_i^{(n-j)}(0) f_i^{(j)}(0) g^j = \sum_{i=1}^{r-1} A_i^{(n)}(0) f_i^{(0)}(0). \tag{5}$$

The induction hypothesis (I) implies that the LHS of (5) is divisible by g^n. Hence we have that

$$f_r^{(n)}(0) - \sum_{i=1}^{r-1}\sum_{j=1}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) \frac{1}{g^{n-j}} A_i^{(n-j)}(0) f_i^{(j)}(0) \in (f_1(0), \cdots, f_{r-1}(0)) : g^n \tag{6}$$

The induction hypothesis (II) implies that the LHS of the equation (6) is also contained in the ideal $(f_1(0), \cdots, f_{r-1}(0)) : h^n$, for any other generic linear form h. Choosing various h's it follows that the LHS of (6) is in fact contained in $(f_1(0), \cdots, f_{r-1}(0)) : (x_1, \cdots, x_{r-1})^n$ by Lemma 2. By Lemma 5 (the duality of a Gorenstein algebra) we see that the following two ideals are equal.

- $(f_1(0), \cdots, f_{r-1}(0)) : (x_1, \cdots, x_{r-1})^n$
- $(f_1(0), \cdots, f_{r-1}(0)) + (x_1, \cdots, x_{r-1})^{d_1+d_2+\cdots+d_{r-1}-(r-1)+1-n}$

But the degree of the LHS of the equation (6) is $d_r - n$ which is strictly smaller than $d_1 + \cdots + d_{r-1} - (r-1) + 1 - n$. This implies that the LHS of (6) lies in fact in the ideal $(f_1(0), \cdots, f_{r-1}(0))$. Thus the LHS of the equation (5) is an element of $g^n(f_1(0), \cdots, f_{r-1}(0))$. This means that the RHS of the equation (5) can be written as

$$\sum_{i=1}^{r-1} A_i^{(n)}(0) f_i^{(0)}(0) = \sum_{i=1}^{r-1} g^n P_i f_i(0),$$

with suitable homogeneous elements $P_i \in k[x_1, \cdots, x_{r-1}]$. Thus $s = (g^n P_i - A_i(0))$ is a syzygy of $((f_1(0), \cdots, f_{r-1}(0))$. By Lemma 4 it is possible to find a syzygy $S = (S_i(\alpha))$ of $(f_i(g\alpha))$ such that $S_i^n(0) = g^n P_i - A_i^n(0)$ and $S_i^{(m)}(0) = 0$, for $m < n$. Now put

$$\tilde{A}_i(\alpha) = A_i(\alpha) + S_i(\alpha)$$

for $i = 1, 2, \cdots, r-1$. It is easy to see that the elements $\tilde{A}_i(\alpha)$ have the desired properties.

4 Weak Lefschetz condition

Let $A = \bigoplus_{i=0}^{c} A_i$ be an Artinian graded algebra over a field A_0. We say that it satisfies the weak Lefschetz condition if there is a linear form $\ell \in A_1$ such that the multiplication map $\cdot\ell : A_i \to A_{i+1}$ is either injective or surjective for each degree i.

Some algebras satisfying this condition were studied in [4] and [6].

Proposition 11 *Let $R = k[x_1, \cdots, x_r]$ be the polynomial ring over a field k of characteristic 0. Let $A = R/(f_1, \cdots, f_r)$, with $d_i = deg\ f_i$, and $2 \leq d_1 \leq \cdots \leq d_r$. If $d_r \geq d_1 + \cdots, d_{r-1} - r$, then A satisfies the weak Lefschetz condition.*

Proof. We indicate an idea of proof. Details are omitted. Put $s(A) = \mathrm{Max}\{\dim A_i \mid i\}$ and $r(A) = \dim A/\ell A$ for a general linear form ℓ. It is easy to see that A satisfies the weak Lefschetz condition if and only if $s(A) = r(A)$. Put $\delta = d_1 + d_2 + \cdots d_{r-1} - (r-1)$ and $m = d_1 d_2 \cdots d_{r-1}$. It is easy to see the following.

$$s(A) = \begin{cases} m, & \text{if } d_r > \delta \\ m-1, & \text{if } d_r = \delta \\ m-2, & \text{if } d_r = \delta - 1 \end{cases}$$

We also have the following.

$$r(A) = \begin{cases} m, & \text{if } d_r > \delta \\ m-1, & \text{if } d_r = \delta \\ m-2, & \text{if } d_r = \delta - 1 \end{cases}$$

We can prove this, in view of Theorem 6, by considering the dimension of the ideal generated by the element f_r in the algebra $R/(f_1, f_2, \cdots, f_{r-1}, \ell)$, where ℓ is a general linear form.

Hence we have $s(A) = r(A)$ under the assumption of Proposition 11.

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