Proc. Sch. Sci. TOKAI UNIV.
48 (2013) 1–9

Power of Certain Multivariate Normal Tests Based on Likelihood Ratio Test with Unknown Covariance Matrix

by

Tsunehisa IMADA[*1] and Hideyuki DOUKE[*2]

[*1] *Department of Business Management, Tokai University*
Toroku, Higashi-ku, Kumamoto, 862-8652, Japan
E-mail: timada@ktmail.tokai-u.jp
[*2] *Department of Mathematical Sciences, Tokai University*
Hiratsuka, Kanagawa, 259-1292, Japan
E-mail: douke@tokai-u.jp
(Received on Sept. 10, 2012; accepted on Dec. 3, 2012)

Abstract

We discuss the power of a certain multivariate one-sided test and a certain multivariate two-sided test for normal mean vectors based on the likelihood ratio test under the assumption that the covariance matrix is unknown. We calculate the power by using conservative critical values for a specified significance level derived by upper bounds of the distributions of the likelihood ratio test statistics. We give some numerical examples regarding critical values and power of the test.
Key words: F-distribution, Monte Carlo simulation, orthogonal projection

1 Introduction

Multivariate one-sided test for testing whether a normal mean vector having nonnegative components is equal to zero or not is a multivariate analog of the one-sided test for a normal mean. Kudo (1963) [3] derived the distribution of the likelihood ratio test statistic under the null hypothesis when the covariance matrix is known. However, if the covariance matrix is unknown, the distribution of the likelihood ratio test statistic depends on the unknown covariance matrix. Perlman (1969) [4] derived an upper bound of the distribution of the likelihood ratio test statistic for all covariance matrices under the null hypothesis. We can obtain a conservative critical value for a specified significance level by using the upper bound.

On the other hand, it is also difficult to derive the distribution of the likelihood ratio test statistic of the multivariate two-sided test for testing whether a normal mean vector is equal to zero or not under the assumption that all components of the normal mean vector are simultaneously nonnegative or nonpositive and the covariance matrix is unknown. Imada *et al.* (2012) [2] derived an upper bound of the distribution of the likelihood ratio test statistic for all covariance matrices under the null hypothesis intended to obtain a conservative critical value for a specified significance level.

In this study we discuss the power of two multivariate tests for normal mean vectors based on the likelihood ratio test under the assumption that the covariance matrix is unknown. We calculate the power by using conservative critical values for a specified significance level derived by upper bounds of the distributions of the likelihood ratio test statistics. We give some numerical examples regarding critical values and power of the test.

2 Multivariate one-sided test

First, we consider a multivariate one-sided test based on the likelihood ratio test under the assumption that the covariance matrix is unknown. Specifically, we discuss an upper bound of the distribution of the likelihood ratio test statistic for all covariance matrices under the null hypothesis derived by Perlman (1969) [4]. Then we consider how to calculate the power of the test for a specified covariance matrix by using Monte Carlo simulation.

2.1 Likelihood ratio test

Assume p-dimensional random variable vector $\boldsymbol{X}$ is distributed according to multivariate normal $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where a mean vector $\boldsymbol{\mu}$ and a covariance matrix $\boldsymbol{\Sigma}$ are unknown. Letting $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$, assume

$$\mu_1 \geq 0, \ \mu_2 \geq 0, \ \ldots, \ \mu_p \geq 0. \tag{2.1}$$

If $\mu_i > 0$ for at least one i $(1 \leq i \leq p)$ in (2.1), we express it as $\boldsymbol{\mu} \geq^* \mathbf{0}$. We set up a null hypothesis H_0 and its alternative hypothesis H_1 as

$$H_0 : \boldsymbol{\mu} = \mathbf{0} \ \text{ vs. } \ H_1 : \boldsymbol{\mu} \geq^* \mathbf{0}. \tag{2.2}$$

We test (2.2) by using a sample $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

By the likelihood ratio test criteria for (2.2) we reject H_0 when

$$\bar{F}_u = \frac{\| \pi_{\boldsymbol{S}}(\boldsymbol{Z}, \boldsymbol{\Theta}_1) \|_{\boldsymbol{S}}^2}{1 + \| \boldsymbol{Z} - \pi_{\boldsymbol{S}}(\boldsymbol{Z}, \boldsymbol{\Theta}_1) \|_{\boldsymbol{S}}^2} > c$$

for a specified critical value c. Here

$$\boldsymbol{Z} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \boldsymbol{X}_i, \quad \bar{\boldsymbol{X}} = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{X}_i, \quad \boldsymbol{S} = \sum_{i=1}^{n} (\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})'$$

and $\pi_{\boldsymbol{S}}(\boldsymbol{Z}, \boldsymbol{\Theta}_1)$ is the orthogonal projection of $\boldsymbol{Z}$ onto

$$\boldsymbol{\Theta}_1 = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_p)' \in \boldsymbol{R}^p | x_1 \geq 0, x_2 \geq 0, \ldots, x_p \geq 0\}$$

with regard to Mahalanobis distance $\| \cdot \|_{\boldsymbol{S}}$ based on $\boldsymbol{S}$. Specifically, $\pi_{\boldsymbol{S}}(\boldsymbol{Z}, \boldsymbol{\Theta}_1)$ is the nearest point in $\boldsymbol{\Theta}_1$ from $\boldsymbol{Z}$. The detailed explanation of the derivation of the likelihood ratio test statistic $\bar{F}_u$ is given by Imada *et al.* (2012) [2].

2.2 Determination of conservative critical value

Since the distribution of $\bar{F}_u$ depends on the unknown $\boldsymbol{\Sigma}$ under H_0, we can not determine the critical value c satisfying

$$P(\bar{F}_u > c) = \alpha$$

under H_0 for a specified significance level α independently of $\boldsymbol{\Sigma}$. However, we can obtain a conservative critical value by using the upper bound of the distribution of $\bar{F}_u$ for all covariance matrices under H_0 derived by Perlman (1969) [4]. The bound is expressed by using F-distributions. Specifically,

$$\max_{\boldsymbol{\Sigma}} P(\bar{F}_u > c) = \frac{1}{2}P(((p-1)/(n-p))F_{p-1,n-p} > c) + \frac{1}{2}P((p/(n-p))F_{p,n-p} > c).$$

If we determine c so that

$$\frac{1}{2}P(((p-1)/(n-p))F_{p-1,n-p} > c) + \frac{1}{2}P((p/(n-p))F_{p,n-p} > c) = \alpha, \tag{2.3}$$

we obtain

$$P(\bar{F}_u > c) \leq \alpha,$$

which shows c is a conservative critical value.

2.3 Calculation of the power of the test

Next, we consider the power of the test. Assuming that p-dimensional vector $\boldsymbol{\mu}_1$ satisfies $\boldsymbol{\mu}_1 \geq^* \mathbf{0}$, we consider the power of the test under the alternative hypotheses $H_1 : \boldsymbol{\mu} = \boldsymbol{\mu}_1$. The power is $P(\bar{F}_u > c)$ under H_1. Since it is difficult to determine the distribution of $\bar{F}_u$ under H_1, we calculate $P(\bar{F}_u > c)$ under H_1 for a specified $\boldsymbol{\Sigma}$ by Monte Carlo simulation as follows.

Specify $\boldsymbol{\Sigma}$ and the number of repetition of the experiment N in advance. Let $R = 1$ and $R_1 = 0$.

Step 1. Generate random number $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ from $N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$.

Step 2. Determine $\bar{F}_u$ by using $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$. Then

$$\begin{aligned} \bar{F}_u > c \quad &\Rightarrow \quad R_1 = R_1 + 1, \\ \bar{F}_u \leq c \quad &\Rightarrow \quad R_1 = R_1. \end{aligned}$$

Step 3. $R = R + 1$. If $R \leq N$, go to Step 1. Otherwise, stop the experiment and let

$$P^* = \frac{R_1}{N}$$

be an approximate value of $P(\bar{F}_u > c)$.

3 Multivariate two-sided test

Next, we consider a multivariate two-sided test based on the likelihood ratio test under the assumption that the covariance matrix is unknown. We discuss an upper bound of the distribution of the likelihood ratio test statistic for all covariance matrices under the null hypothesis derived by Imada *et al.* (2012) [2] and the power of the test.

3.1 Likelihood ratio test

Assume p-dimensional random variable vector $\boldsymbol{X}$ is distributed according to multivariate normal $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown. Letting $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$, assume

$$\mu_1 \geq 0,\ \mu_2 \geq 0,\ \ldots,\ \mu_p \geq 0$$

or

$$\mu_1 \leq 0,\ \mu_2 \leq 0,\ \ldots,\ \mu_p \leq 0. \tag{3.1}$$

If $\mu_i < 0$ for at least one $i (1 \leq i \leq p)$ in (3.1), we express it as $\boldsymbol{\mu} \leq^* \mathbf{0}$. We set up a null hypothesis H_0 and its alternative hypothesis H_1 as

$$H_0 : \boldsymbol{\mu} = \mathbf{0} \ \text{ vs. } \ H_1 : \boldsymbol{\mu} \geq^* \mathbf{0} \ \text{ or } \ \boldsymbol{\mu} \leq^* \mathbf{0}. \tag{3.2}$$

We call the test (3.2) a multivariate two-sided test. We test (3.2) by using a sample $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

We summarize the likelihood ratio test for (3.2) according to Imada *et al.* (2012) [2]. By the likelihood ratio test criteria for (3.2) we reject H_0 when

$$\bar{\bar{F}} = \frac{\| \pi_{\boldsymbol{S}}(\boldsymbol{Z}, \boldsymbol{\Theta}) \|^2_{\boldsymbol{S}}}{1 + \| \boldsymbol{Z} - \pi_{\boldsymbol{S}}(\boldsymbol{Z}, \boldsymbol{\Theta}) \|^2_{\boldsymbol{S}}} > c$$

for a specified critical value c. Here

$$\boldsymbol{\Theta} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_p)' \in \boldsymbol{R}^p | x_1 \geq 0, x_2 \geq 0, \ldots, x_p \geq 0 \text{ or } x_1 \leq 0, x_2 \leq 0, \ldots, x_p \leq 0\}.$$

Then

$$\bar{\bar{F}} = \max\{\bar{F}_u, \bar{F}_l\}$$

where

$$\bar{F}_l = \frac{\| \pi_{\boldsymbol{S}}(\boldsymbol{Z}, \boldsymbol{\Theta}_2) \|^2_{\boldsymbol{S}}}{1 + \| \boldsymbol{Z} - \pi_{\boldsymbol{S}}(\boldsymbol{Z}, \boldsymbol{\Theta}_2) \|^2_{\boldsymbol{S}}}$$

and

$$\boldsymbol{\Theta}_2 = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_p)' \in \boldsymbol{R}^p | x_1 \leq 0, x_2 \leq 0, \ldots, x_p \leq 0\}.$$

Note that $\bar{F}_l$ is the likelihood ratio test statistic for the multivariate lower-sided test

$$H_0 : \boldsymbol{\mu} = \mathbf{0} \ \text{ vs. } \ H_1 : \boldsymbol{\mu} \leq^* \mathbf{0}.$$

3.2 Determination of conservative critical value

Although it is difficult to determine the distribution of $\bar{\bar{F}}$, Imada *et al.* (2012) [2] derived

$$\max_{\boldsymbol{\Sigma}} P(\bar{\bar{F}} > c) \leq P(((p-1)/(n-p))F_{p-1,n-p} > c) + P(((p/(n-p))F_{p,n-p} > c).$$

If we determine c so that

$$P(((p-1)/(n-p))F_{p-1,n-p} > c) + P(((p/(n-p))F_{p,n-p} > c) = \alpha, \tag{3.3}$$

we obtain

$$P(\bar{\bar{F}} > c) \leq \alpha,$$

which shows c is a conservative critical value for (3.2).

3.3 Calculation of the power of the test

Next, we consider the power of the test. Assuming that p-dimensional vector $\boldsymbol{\mu}_1$ satisfies $\boldsymbol{\mu}_1 \geq^* \mathbf{0}$

or $\boldsymbol{\mu}_1 \leq^* \mathbf{0}$, the power of the test under the alternative hypotheses $H_1 : \boldsymbol{\mu} = \boldsymbol{\mu}_1$ is given by $P(\bar{\bar{F}} > c)$ under H_1. It can be calculated by Monte Carlo simulation. The method is similar to that discussed in subsection 2.3.

4 Numerical examples

In this section we give some numerical examples regarding conservation of critical values and the power of the test for the multivariate one-sided test and the multivariate two-sided test. Numerical results regarding conservation of critical values were given by Imada *et al.* (2012) [2]. We refer to them.

First, let $p = 2$. Let $\alpha = 0.05$ and $n = 20, 30$. By (2.3) we obtain conservative critical values for multivariate one-sided test $c = 0.334, 0.203$ for $n = 20, 30$, respectively. By (3.3) we obtain conservative critical values for multivariate two-sided test $c = 0.437, 0.262$ for $n = 20, 30$, respectively. We give Type I error of two tests for specified $\boldsymbol{\Sigma}$. Specifically,

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

where $\rho = -0.9, -0.6, -0.3, 0, 0.3, 0.6, 0.9$. Tables 1 and 2 give Type I error of two tests obtained by Monte Carlo simulation with 100,000 random numbers. (All numerical results by Monte Carlo simulation in this paper are calculated by 100,000 random numbers.) Tables show that the critical value is less conservative as ρ is larger from -0.9 to 0.9 for two tests. The critical value of multivariate two-sided test is uniformly more conservative compared to that of multivariate one-sided test. The difference of Type I error between two tests is larger as ρ is larger from -0.9 to 0.9 for two tests. Type I error does not vary remarkably as the sample size n varies from 20 to 30 for two tests.

Table 1: Type I error of multivariate one-sided test by using conservative critical values ($p = 2$)

ρ	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
$n = 20$	0.0170	0.0234	0.0274	0.0307	0.0353	0.0387	0.0437
$n = 30$	0.0171	0.0226	0.0276	0.0308	0.0344	0.0385	0.0441

Table 2: Type I error of multivariate two-sided test by using conservative critical values ($p = 2$)

ρ	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
$n = 20$	0.0170	0.0217	0.0268	0.0310	0.0339	0.0358	0.0372
$n = 30$	0.0170	0.0222	0.0267	0.0307	0.0337	0.0361	0.0384

Next, we consider the power of the test. We calculate the power for two tests by using critical values given in Tables 1, 2 under the alternative hypotheses $H_1 : \boldsymbol{\mu} = \boldsymbol{\mu}_1$ where $\boldsymbol{\mu}_1 = (0.5, 0.5)', (0.5, 0.25)', (0.5, 0)'$. Table 3 gives the power of multivariate one-sided test. Table 4 gives the power of multivariate two-sided test. They show that the variations of the power of two tests corresponding to the variation of ρ for each H_1 are similar. Specifically, under $H_1 : \boldsymbol{\mu} = (0.5, 0.5)'$ the power decreases as ρ increases from -0.9 to 0.9. Under $H_1 : \boldsymbol{\mu} = (0.5, 0.25)'$ and $H_1 : \boldsymbol{\mu} = (0.5, 0)'$ the power decreases as ρ increases from -0.9 to a certain value, then the power increases as ρ increases. From the results we can not specify the value of ρ giving the minimum power for $H_1 : \boldsymbol{\mu} = (0.5, 0.25)'$ and $H_1 : \boldsymbol{\mu} = (0.5, 0)'$. Although the critical value is less conservative as ρ is larger, the variation of the power does not correspond to that. The power with $n = 30$ is uniformly higher than that with $n = 20$ for each test, although the difference of Type I error is small between two cases.

Table 3: Power of multivariate one-sided test ($p = 2$)

	ρ	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
$\boldsymbol{\mu} = (0.5, 0.5)'$	$n = 20$	1.000	0.994	0.920	0.803	0.696	0.604	0.537
	$n = 30$	1.000	1.000	0.990	0.945	0.876	0.801	0.732
$\boldsymbol{\mu} = (0.5, 0.25)'$	$n = 20$	1.000	0.920	0.733	0.601	0.529	0.517	0.767
	$n = 30$	1.000	0.989	0.904	0.798	0.724	0.712	0.926
$\boldsymbol{\mu} = (0.5, 0)'$	$n = 20$	0.993	0.658	0.502	0.464	0.505	0.658	0.993
	$n = 30$	1.000	0.856	0.708	0.665	0.707	0.854	1.000

Table 4: Power of multivariate two-sided test ($p = 2$)

	ρ	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
$\boldsymbol{\mu} = (0.5, 0.5)'$	$n = 20$	1.000	0.984	0.853	0.696	0.569	0.476	0.406
	$n = 30$	1.000	1.000	0.975	0.896	0.794	0.700	0.617
$\boldsymbol{\mu} = (0.5, 0.25)'$	$n = 20$	1.000	0.855	0.615	0.471	0.400	0.389	0.655
	$n = 30$	1.000	0.976	0.837	0.694	0.609	0.594	0.868
$\boldsymbol{\mu} = (0.5, 0)'$	$n = 20$	0.982	0.531	0.377	0.344	0.388	0.550	0.986
	$n = 30$	1.000	0.772	0.597	0.549	0.601	0.782	1.000

Next, let $p = 3$. Let $\alpha = 0.05$ and $n = 20, 30$. By (2.3) we obtain conservative critical values for multivariate one-sided test $c = 0.503, 0.295$ for $n = 20, 30$, respectively. By (3.3) we obtain conservative critical values for multivariate two-sided test $c = 0.638, 0.367$ for $n = 20, 30$, respectively. Tables 5 and 6 give Type I error for two tests by using conservative critical values for

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$$

with $\rho = -0.3, 0, 0.3, 0.6, 0.9$. The results are similar to those for $p = 2$.

Next, we consider the power of the test. We calculate the power for two tests by using critical values given in Tables 5, 6 under the alternative hypotheses $H_1 : \boldsymbol{\mu} = \boldsymbol{\mu}_1$ where $\boldsymbol{\mu}_1 = (0.5, 0.5, 0.5)'$, $(0.5, 0.5, 0.25)'$, $(0.5, 0.5, 0)'$, $(0.5, 0.25, 0)'$, $(0.5, 0, 0)'$. Table 7 gives the power of multivariate one-sided test. Table 8 gives the power of multivariate two-sided test. They show that the variations

Table 5: Type I error of multivariate one-sided test by using conservative critical values ($p = 3$)

ρ	-0.3	0	0.3	0.6	0.9
$n = 20$	0.0139	0.0215	0.0271	0.0342	0.0435
$n = 30$	0.0140	0.0212	0.0270	0.0348	0.0421

Table 6: Type I error of multivariate two-sided test by using conservative critical values ($p = 3$)

ρ	-0.3	0	0.3	0.6	0.9
$n = 20$	0.0141	0.0199	0.0275	0.0322	0.0339
$n = 30$	0.0133	0.0206	0.0258	0.0317	0.0348

of the power of two tests corresponding to the variation of ρ for each H_1 are similar. Specifically, under $H_1 : \boldsymbol{\mu} = (0.5, 0.5, 0.5)'$ the power decreases as ρ increases from -0.3 to 0.9. Under other alternative hypotheses the power decreases as ρ increases from -0.3 to a certain value, then the power increases as ρ increases. From the results we can not specify the value of ρ giving the minimum power in those cases. For each test the power with $n = 30$ is uniformly higher than that with $n = 20$. The aspect of variation of the power does not correspond to that of Type I error.

Table 7: Power of multivariate one-sided test ($p = 3$)

	ρ	-0.3	0	0.3	0.6	0.9
$\boldsymbol{\mu} = (0.5, 0.5, 0.5)'$	$n = 20$	0.999	0.875	0.680	0.541	0.442
	$n = 30$	1.000	0.980	0.881	0.755	0.646
$\boldsymbol{\mu} = (0.5, 0.5, 0.25)'$	$n = 20$	0.986	0.754	0.582	0.532	0.786
	$n = 30$	1.000	0.930	0.806	0.753	0.947
$\boldsymbol{\mu} = (0.5, 0.5, 0)'$	$n = 20$	0.919	0.666	0.618	0.730	0.997
	$n = 30$	0.992	0.878	0.840	0.920	1.000
$\boldsymbol{\mu} = (0.5, 0.25, 0)'$	$n = 20$	0.679	0.435	0.421	0.551	0.980
	$n = 30$	0.880	0.654	0.635	0.780	0.999
$\boldsymbol{\mu} = (0.5, 0, 0)'$	$n = 20$	0.330	0.271	0.331	0.523	0.991
	$n = 30$	0.490	0.419	0.505	0.725	1.000

Table 8: Power of multivariate two-sided test ($p = 3$)

	ρ	-0.3	0	0.3	0.6	0.9
$\boldsymbol{\mu} = (0.5, 0.5, 0.5)'$	$n = 20$	0.996	0.784	0.552	0.405	0.317
	$n = 30$	1.000	0.957	0.802	0.643	0.521
$\boldsymbol{\mu} = (0.5, 0.5, 0.25)'$	$n = 20$	0.967	0.636	0.450	0.400	0.675
	$n = 30$	0.999	0.874	0.702	0.641	0.898
$\boldsymbol{\mu} = (0.5, 0.5, 0)'$	$n = 20$	0.855	0.536	0.488	0.618	0.994
	$n = 30$	0.983	0.803	0.751	0.867	1.000
$\boldsymbol{\mu} = (0.5, 0.25, 0)'$	$n = 20$	0.566	0.321	0.311	0.441	0.964
	$n = 30$	0.826	0.543	0.526	0.694	0.999
$\boldsymbol{\mu} = (0.5, 0, 0)'$	$n = 20$	0.244	0.191	0.256	0.474	0.925
	$n = 30$	0.413	0.331	0.433	0.718	0.928

5 Conclusions

In this study we have discussed the multivariate one-sided test and the multivariate two-sided test for normal mean vectors based on the likelihood ratio test under the assumption that the covariance matrix is unknown. Specifically, we considered the power of the test by using the conservative critical values derived by the upper bounds of the likelihood ratio test statistics. We gave some numerical examples regarding critical values and power of the test. We confirmed that the aspect of variation of power does not correspond to that of conservation of the critical value. We should investigate causes of those phenomenons.

On the other hand, Fraser, Guttman and Srivastava (1991) [1] constructed another kind of critical value of the multivariate one-sided test based on the likelihood ratio test. Their critical value satisfies a specified significance level exactly under the condition that the orthogonal projection is determined. It will be possible to construct a conservative critical value of the multivariate two-sided test by referring to their idea. We will give numerical examples regarding Type I error and the power for these tests intended to compare them with the tests discussed in this study in the future.

Acknowledgements

The authors are deeply grateful to the referee and the editors for their valuable comments and suggestions.

References

[1] Fraser, D. A. S., Guttman, I. and Srivastava, M. S. (1991). Conditional inference for treatment and error in multivariate analysis. Biometrika 78, 3, 565-572.

[2] Imada, T., Douke, H. and Fujii, I. (2012). Conservative Critical Value of Certain Multivariate Tests for Normal Mean Vectors Based on Likelihood Ratio Test with Unknown Covariance Matrix.

Proceedings of The School of Science of Tokai University 47, 11-20.

[3] Kudo, A. (1963). A Multivariate Analogue of the One-sided Test, Biometrika 50, 403-418.

[4] Perlman, M. D. (1969). One-Sided Testing Problems in Multivariate Analysis, The Annals of Mathematical Statistics 40, 549-567.